Gauge-invariant coherent states for Loop Quantum Gravity I: Abelian gauge groups

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February 2, 2008

Abstract

In this paper we investigate the properties of gauge-invariant coherent states for Loop Quantum Gravity, for the gauge group $U(1)$. This is done by projecting the corresponding complexifier coherent states, which have been applied in numerous occasions to investigate the semiclassical limit of the kinematical sector, to the gauge-invariant Hilbert space. This being the first step to construct *physical* coherent states, we arrive at a set of gauge-invariant states that approximate well the gauge-invariant degrees of freedom of abelian LQG. Furthermore, these states turn out to encode explicit information about the graph topology, and show the same pleasant peakedness properties known from the gauge-variant complexifier coherent states.

1 Introduction

Loop Quantum Gravity (LQG) is a promising candidate for a theory that aims to combine the principles of quantum mechanics and general relativity (see [1, 2, 3, 4] and references therein). The starting point of LQG is the Hamiltonian formulation of general relativity, choosing Ashtekar-variables as phase-space coordinates, which casts GR into a $SU(2)$ gauge theory, leading to the Poisson structure

$$\left\{A_a^I(x)\,,\,A_b^J(y)\right\} = \left\{E_I^a(x)\,,\,E_J^b(y)\right\} \;=\; 0 \tag{1.1}$$

$$\left\{A_a^I(x)\,,\,E_J^b(y)\right\} = 8\pi G\beta\;\delta_b^a\,\delta_J^I\;\delta(x-y). \tag{1.2}$$

This system could be canonically quantized with the help of methods well-known from algebraic quantum field theory, which resulted in a representation of the Poisson-algebra on a Hilbert space $\mathcal{H}_{kin}$, which carries the kinematical information of quantum general relativity. One has found recently [5] that this representation is unique up to unitary equivalence if one demands the space-diffeomorphisms to be unitarily implemented.

While the dynamics of classical general relativity is encoded into a set of phase-space functions G_I, D_a, H that are constrained to vanish, these so-called constraints are, in LQG, promoted to operators that generate gauge-transformations on the kinematical Hilbert space $\mathcal{H}_{kin}$. The physical Hilbert space $\mathcal{H}_{phys}$ is then to be derived as the set of (generalized) vectors being invariant under these gauge-transformations [6].

$$\hat{G}_I|\psi\rangle \;=\; \hat{D}_a|\psi\rangle \;=\; \hat{H}|\psi\rangle \;=\; 0. \tag{1.3}$$

Although conceptually clear, the actual computation of $\mathcal{H}_{phys}$ is technically quite difficult. This is due to the fact that the constraints $\hat{G}_I$, $\hat{D}_a$ $\hat{H}$ act quite non-trivially on $\mathcal{H}_{kin}$. Thus, while the kinematical setting is understood, the physical states of the theory are not known explicitly. It seems that, in its present formulation, LQG is too complicated to be solved analytically.

While this seems to be discouraging at first, complete solvability is not something one could have expected from the outset. In fact, nearly no theory which realistically describes a part of nature is completely solvable, neither in the quantum, nor in the classical regime. Rather, having the basic equations of a theory as a starting point, one has to develop tools for extracting knowledge about its properties in special cases, reducing the theory to simpler subsectors, approximate some solutions of the theory, or study its behavior via numerical methods. Examples for this range from reducing classical GR to symmetry-reduced situations, which is our main source of understanding the large-scale structure of our cosmos, over particle physics, where perturbative quantum field theory is our access to predict the behavior of elementary particles, to numerical simulations in ordinary quantum mechanics, which allow for computations of atomic and molecular spectra, transition amplitudes or band structures in solid state physics. Although in all of these fields the fundamental equations are well-known, their complete solution is elusive, so one has to rely on approximations and numerics in order to understand the physical processes described by them. In other cases, such as interacting Wightman fields on 4D Minkowski space, not a single example is known to date. On the other hand, the perturbation theory for, say, $SU(N)$-Yang-Mills theory in small couplings is so effective that many particle physicists even regard the perturbative expansion in the coupling parameter as the fundamental theory in itself.

With these considerations, it seems quite natural to look for a way to gain knowledge about the physical content of LQG by approximation methods. One step into this direction has been done by introducing the complexifier coherent states.

For ordinary quantum mechanics, the well-known harmonic oscillator coherent

states (HOCS)

$$|z\rangle \;=\; \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\,|n\rangle \tag{1.4}$$

are a major tool for performing analytical calculations and numerical computations. Not only can they be used to approximate quantum propagators [7], they are also the main tool for investigating the transition from quantum to classical behaviour, as well as quantum chaos [8, 9]. They also grant access to the numerical treatment of quantum dynamics for various systems [10, 11], and their generalization to quantum electrodynamics provides a path to the accurate description of laser light and quantum optics [12].

The complexifier coherent states (CCS), which have been first introduced in [13, 14], are a natural generalization of the HOCS to quantum mechanics on cotangent bundles over arbitrary compact Lie groups, and the complexifier methods employed to construct these states can also be transferred to other manifolds as well. Furthermore, for the special cases of quantum mechanics on the real line $\mathbb{R}$ and the circle $U(1)$, these states reduce to what has been used as coherent states for quite some time [15, 16].

In [17], the complexifier concept has been used to define complexifier coherent states for LQG. They are states on the kinematical Hilbert space $\mathcal{H}_{kin}$ and their properties have been exhibited in [18, 19]. It was shown that they mimic the HOCS in their semiclassical behavior, in the sense that they describe the quantum system to be close to some point in the corresponding classical phase-space of general relativity, minimizing relative fluctuations. Also, they provide a Bargman-Segal representation of $\mathcal{H}_{kin}$ as holomorphic functions, as well as approximating well quantum observables that correspond to classical phase space variables.

This has indicated that these states are a useful tool for examining the semiclassical limit of LQG. In particular, it has been shown [20] with the help of the CCS that the constraint operators for LQG, which are defined on $\mathcal{H}_{kin}$ and generate the dynamics of the theory, have the correct classical limit. In particular, CCS that are "concentrated" around a classical solution of GR, are annihilated by the constraint operators up to orders of $\hbar$. This indicates that, at least infinitesimally, LQG has classical GR as semiclassical limit.

On the other hand, since the complexifier coherent states are only defined on $\mathcal{H}_{kin}$, none of them is really physical in the sense of the Dirac quantization programme. That is, while they are peaked on the classical constraint surface, they are not annihilated by the constraint operators, only approximately. Thus, while being a good tool for examining kinematical properties of LQG, it is not clear how well they approximate the dynamical aspects of quantum general relativity.

To do this, it would be desirable to have coherent states at hand that satisfy at least some of (1.3). We will pursue the first step on this path in this and the following article.

Some of the constraints (1.3) are simpler than others. In particular, the easiest ones are the Gauss constraints $\hat{G}_I$. They are unbounded self-adjoint operators on $\mathcal{H}_{kin}$ and the gauge-transformations generated by them are well understood. The set of vectors being invariant under the Gauss-gauge-transformations ("gauge-invariant" in the following) is a proper subspace of $\mathcal{H}_{kin}$. This space is well known

[21], and a basis for it is provided by the gauge-invariant spin network functions, the construction of which involve intertwiners of the corresponding gauge group $SU(2)$. Thus, the straightforward way to construct gauge-invariant coherent states would be to project the CCS to the gauge-invariant Hilbert space. We will do exactly that in this and the following article.

The gauge transformations correspond to gauging the $\mathfrak{su}(2)$-valued Ashtekar connection A_a^I and its canonically conjugate, the electric flux E_I^a. Thus, the gauge group $SU(2)$ is involved, and in fact this group plays a prominent role in the construction of the whole kinematical Hilbert space $\mathcal{H}_{kin}$. It is, however, possible to replace $SU(2)$ in this construction by any compact gauge group G, arriving at a different kinematical Hilbert space $\mathcal{H}_{kin}^G$, which would be the arena for the Hamiltonian formulation of a gauge field theory with gauge group G. Of course, one also has to replace the $\hat{G}_I$ by the corresponding gauge generators. Also the constraints $\hat{D}_a$ and $\hat{H}$ can, although nontrivial, be modified to match the new gauge group. Finally, the complexifier method is able to supply corresponding coherent states for each gauge group G.

This change of $SU(2)$ into another gauge group has been used frequently. In [22] it has been shown that the quantization of linearized gravity leads to the LQG framework with $U(1)^3$ as gauge group. Furthermore, it has been pointed out [23] that changing $SU(2)$ for $U(1)^3$ does not change the qualitative behavior of the theory in the semiclassical limit, and so the $U(1)^3$-CCS have been used widely in order to investigate LQG [20].

Before treating the much more complicated case of $G = SU(2)$ in [24], in this paper we will, as a warm-up, consider the gauge group $G = U(1)$ and the corresponding CCS. The case $G = U(1)^3$ is then simply obtained by a triple tensor product: Not only the kinematical Hilbert space

$$\mathcal{H}_{kin}^{U(1)^3} \;=\; \mathcal{H}_{kin}^{U(1)} \,\otimes\, \mathcal{H}_{kin}^{U(1)} \,\otimes\, \mathcal{H}_{kin}^{U(1)} \tag{1.5}$$

has this simple product structure, but also the respective gauge-invariant subspaces decompose according to (1.5). Also, $U(1)^3$-CCS are obtained by tensoring three $U(1)$-CCS. Due to this simple structure, it is sufficient for our arguments to consider the gauge-invariant coherent states in the case of $G = U(1)$, since all the properties revealed in this article can be carried over straightforwardly to gauge-invariant coherent states for $G = U(1)^3$.

The plan for this paper is as follows: In chapter 2, we will shortly repeat the basics of LQG. In particular, the kinematical Hilbert space $\mathcal{H}_{kin}$ for arbitrary gauge group G is defined, the corresponding set of constraints that generate the gauge-transformations are described. In chapter 3, the complexifier coherent states are defined, where the focus lies on the particular case of $G = U(1)$. A formula for the inner product between two such states is derived, which depends purely on the geometry of the complexification of the gauge group $U(1)^{\mathbb{C}} \simeq \mathbb{C}\backslash\{0\}$. Although this is not of particular importance in this article, we will find a similar formula in [24], when we come to the case of $G = SU(2)$. This will hint towards a geometric interpretation of the CCS for arbitrary gauge groups, and we will comment shortly on this at the end of [24].

In chapter 4 we will apply the projector onto the gauge-invariant subspace of $\mathcal{H}_{kin}$ to the $U(1)$-complexifier coherent states. The involved gauge integrals can be carried out by a special procedure resembling a gauge-fixing. The resulting gauge-invariant states are then investigated, and their properties are displayed. In particular, we will show that they describe semiclassical states peaked at gauge-invariant degrees of freedom.

We will conclude this article with a summary and an outlook to the sequel paper.

2 The kinematical setting of LQG

In this section, we will shortly repeat the kinematical framework of LQG.

Loop Quantum Gravity is a quantization of a Hamiltonian formulation of classical GR. This is done by introducing an ADM split of space-time and the introduction of Ashtekar variables [1]. Thus, GR can be formulated as a constrained SU(2)-gauge theory on a tree-dimensional manifold Σ, which is regarded as space, and is taken to be compact. The quantization for noncompact Σ can also be carried out, but this requires some more mathematical effort.

On Σ the Ashtekar $\mathfrak{su}(2)$-connection A_a^I and the electric flux E_I^a are the dynamical variables. They are canonically conjugate to each other:

$$\begin{aligned}\left\{A_a^I(x)\,,\,A_b^J(y)\right\} &= \left\{E_I^a(x)\,,\,E_J^b(y)\right\} \;=\; 0\\ \left\{A_a^I(x)\,,\,E_J^b(y)\right\} &= 8\pi G\beta\;\delta_b^a\,\delta_J^I\;\delta(x-y).\end{aligned}$$

The fields are not free, but subject to so-called constraints, which are phase-space functions, i.e. functions of A and E. They encode the diffeomorphism-invariance of the theory, and the Einstein equations. The reduced phase space consists of all phase space points A, E where the constraints vanish. On this set, the constraints act as gauge transformations, and the set of gauge orbits is the physical phase space. The set of constraints is divided into the Gauss constraints $G_I(x)$, the diffeomorphism constraints $D_a(x)$ and the Hamilton constraints $H(x)$. These satisfy the Poisson algebra

$$\begin{aligned}\Big\{G(s),\,G(t)\Big\} &= G(s\wedge t)\\ \Big\{G(s),\,D(f)\Big\} &= \Big\{G(s),\,H(g)\Big\} \;=\; 0\\ \Big\{D(f),\,D(g)\Big\} &= D(\mathcal{L}_f g)\\ \Big\{D(f),\,H(n)\Big\} &= H(\mathcal{L}_f n)\\ \Big\{H(n),\,H(m)\Big\} &= D(g^{ab}(n\,m,_b - m\,n,_b))\end{aligned} \tag{2.1}$$

where s,t are $\mathfrak{su}(2)$-valued functions, f,g are vector fields on Σ, n,m are scalar functions on Σ, the smeared constraints are defined by

$$G(s) \ := \ \int_\Sigma G_I(x) s^I(x), \qquad D(f) \ := \ \int_\Sigma D_a(x)\, f^a(x), \qquad H(n) \ := \ \int_\Sigma H(x)\, n(x),$$

d denotes the exterior derivative on Σ, $\mathcal{L}$ the Lie derivative, and $\flat$ is the isomorphism from one-forms to vector fields provided by the metric. It is this particular occurrence of the metric itself in the Poisson brackets, which makes the algebra structure notoriously difficult.

2.1 The kinematical Hilbert space

The kinematical Hilbert space $\mathcal{H}_{kin}$ of LQG is computed as a directed limit of Hilbert spaces of functions being cylindrical over a particular graph embedded in Σ. Consider γ to be a graph, consisting of finitely many oriented edges $e_1, \ldots, e_E$ being embedded analytically in Σ, such that the intersection of two edges is either empty or a common endpoint, or vertex v. For each such graph γ there is a Hilbert space H_γ, which consists of all functions being cylindrical over that particular γ. In particular, each edge E of the graph defines a function from the set of all connections

$$h_e : \mathcal{A} \longrightarrow SU(2)$$

by setting $h_e(A)$ being the holonomy of the connection A along the edge e. Symbolically,

$$h_e(A) \ = \ \mathcal{P} \exp i \int_0^1 dt \ A_a^I(e(t)) \frac{\tau_I}{2} \, \dot{e}^a(t).$$

A function $f : \mathcal{A} \to \mathbb{C}$ is cylindrical over the graph γ, having E edges $e_1, \ldots e_E$ if there is a function $\tilde{f} : SU(2)^E \to \mathbb{C}$ with

$$f(A) \ = \ \tilde{f}\Big(h_{e_1}(A), \, \ldots, \, h_{e_E}(A)\Big). \tag{2.2}$$

The integration measure in this Hilbert space is just the Haar measure on $SU(2)^E$, which gives the canonical isomorphism

$$H_\gamma \ \simeq \ L^2\Big(SU(2)^E, \, d\mu_H^{\otimes E}\Big). \tag{2.3}$$

The set of graphs is a partially ordered set. Let γ, γ' be two graphs, then one writes $\gamma \preceq \gamma'$, iff there is a subdivision γ'' of γ' by inserting additional vertices into the edges, such that γ is a subgraph of γ''. Note that, since all graphs consist of analytically embedded edges, this indeed defines a partially ordering, i.e. for any two graphs γ_1, γ_2 there is always a γ_3 such that $\gamma_1 \preceq \gamma_3$ and $\gamma_2 \preceq \gamma_3$.

Each function f_γ cylindrical over γ determines a cylindrical function $f_{\gamma''}$ over γ'', simply by defining

$$\tilde{f}_{\gamma''}(h_{e_1}(A), \ldots, h_{e'_E}(A)) \ := \ \tilde{f}_\gamma(h_{e_{n_1}}(A), \ldots, \, h_{e_{n_E}}(A)) \tag{2.4}$$

where $e_{n_1}, \ldots, e_{n_E}$ are the edges in γ'' belonging to γ. Now, every function cylindrical over γ'' is also obviously cylindrical over γ', since γ'' is only a refinement of γ'. This procedure defines a unitary map

$$U_{\gamma\gamma'} \ : \ \mathcal{H}_\gamma \ \longrightarrow \ \mathcal{H}_{\gamma'}.$$

One can show that for $\gamma \preceq \gamma' \preceq \gamma''$, one has $U_{\gamma'\gamma''} U_{\gamma\gamma'} \ = \ U_{\gamma\gamma''}$. So, this family of unitary maps defines a projective limit

$$\mathcal{H}_{kin} \ := \ \varinjlim \ \mathcal{H}_\gamma, \tag{2.5}$$

which serves as the kinematical Hilbert space of LQG. Each $\mathcal{H}_\gamma$ has a canonical isometric embedding U_γ into $\mathcal{H}_{kin}$, which is compatible with the unitary maps $U_{\gamma\gamma'}$ in the following way:

$$U_{\gamma\gamma'} \, U_{\gamma'} \ = \ U_\gamma \qquad \text{for all } \gamma \preceq \gamma'.$$

Due to the definition of the inner product in the projective limit, for $\psi_\gamma \in \mathcal{H}_\gamma$ and $\psi_{\gamma'} \in \mathcal{H}_{\gamma'}$, where the intersection of γ and γ' is empty, one has that

$$\left\langle U_\gamma \psi_\gamma \middle| U_{\gamma'} \psi_{\gamma'} \right\rangle \ = \ 0.$$

This immediately shows that, since there are uncountably many graphs with mutual empty intersection in Σ, $\mathcal{H}_{kin}$ cannot be separable. On the other hand, since $\mathcal{H}_{kin}$ is built up out of the $\mathcal{H}_\gamma$, we can restrict our considerations to an arbitrary but fixed graph γ for most purposes, dealing only with the Hilbert space $\mathcal{H}_\gamma$, which is separable.

Note that the whole construction carried out here can be done with an arbitrary compact Lie group G. The field A is then a connection on a $\mathfrak{g}$-bundle and E the corresponding electric flux, which is canonically conjugate. Also the definition of the constraints can be adapted to build a theory for arbitrary gauge groups. This is not only a mathematical toy, but in some situations, it is in fact useful to replace the gauge group $SU(2)$ by $U(1)^3$, which can be physically justified [19, 22, 23]. In particular, we will deal in this article with the complexifier- and gauge-invariant coherent states for the case of $G = U(1)$, which will serve as a warm-up example before coming to the much more difficult (but also more realistic) case of $G = SU(2)$ in [24].

2.2 Constraint operators and gauge actions

In the previous section the kinematical framework for LQG was presented. In this section, we will shortly discuss the constraint operators and the gauge actions they induce on $\mathcal{H}_{kin}$.

Rewriting general relativity in a Hamiltonian formulation using the Ashtekar variables results in the formulation of the Ashtekar connection $A_a^I(x)$ and the electric flux $E_I^a(x)$, which, in the quantized theory, become operators on $\mathcal{H}_{kin}$. One cannot

quantize the fields directly, but has to smear them with certain test functions having support on one-dimensional and two-dimensional submanifolds of Σ, respectively. See [1] for details.

In the classical theory, the dynamics is encoded in the constraints (2.1), which in the quantum theory become operators acting on $\mathcal{H}_{kin}$. The physical Hilbert space is determined by the condition that (generalized) states are annihilated by the constraint operators

$$\hat{D}_a\,\psi_{phys} \;=\; \hat{G}_I\,\psi_{phys} \;=\; \hat{H}\,\psi_{phys} \;=\; 0. \tag{2.6}$$

To implement the Gauss constraints as operators on Σ is, actually, quite straightforward. Since the kinematical Hilbert space $\mathcal{H}_{kin}$ can be thought of as being built up from $\mathcal{H}_\gamma$ for arbitrary graphs $\gamma\subset\Sigma$ by (2.5), it is sufficient to compute the gauge-transformation generated by the $\hat{G}_I$.

In particular, the similarity between LQG and a lattice gauge theory on γ is displayed, if one computes the unitary group generated by the constraints $\hat{G}_I(x)$, which correspond to $SU(2)$-gauge transformations of functions on the graph. In particular, let $k:\Sigma\to SU(2)$ be a function and f a cylindrical function over a graph γ with E edges. The action of k on f is given by the induced action of k on the corresponding $\tilde{f}:SU(2)^E\to\mathbb{C}$ via (2.2), to be

$$\alpha_k\tilde{f}\,\left(h_{e_1},\ldots,h_{e_E}\right) \;:=\; \tilde{f}\,\left(k_{b(e_1)}h_{e_1}k^{-1}_{f(e_1)},\ldots,k_{b(e_E)}h_{e_E}k^{-1}_{f(e_E)}\right), \tag{2.7}$$

where $b(e_m)$ and $f(e_m)$ are the beginning- and end-point of the edge e_m, and $k_x\in SU(2)$ is the value of the map k at $x\in\Sigma$. So, the gauge transformations act only at the vertices of a graph.

In particular, one can write down the projector onto the gauge-invariant Hilbert space for functions in $\mathcal{H}_\gamma$:

$$\begin{aligned}\mathcal{P}f(h_{e_1},\ldots,h_{e_E}) &:= \int_{SU(2)^V} d\mu_H(k_1,\ldots,k_V)\alpha_{k_1,\ldots k_V}\,f(h_{e_1}\ldots,h_{e_E}) \qquad (2.8)\\ &= \int_{SU(2)^V} d\mu_H(k_1,\ldots,k_V) f\Big(k_{b(e_1)}h_{e_1}k^{-1}_{f(e_1)},\ldots,k_{b(e_E)}h_{e_E}k^{-1}_{f(e_E)}\Big)\end{aligned}$$

Since there are only finitely many vertices on the graph γ, the integral exists and defines a projector

$$\mathcal{P}:\;\mathcal{H}_\gamma\;\longrightarrow\;\mathcal{H}_\gamma$$

onto a sub-Hilbert space of $\mathcal{H}_\gamma$. In particular, the gauge-invariant functions on a graph form a subset of all cylindrical functions on a graph. The gauge-invariant Hilbert spaces can be described using intertwiners between irreducible representations of $SU(2)$, and a basis for the gauge-invariant Hilbert spaces $\mathcal{P}\mathcal{H}_\gamma$ can be written down in terms of gauge-invariant spin network functions [21].

The diffeomorphism constraints $\hat{D}$ can, however, not be implemented as operators on $\mathcal{H}_{kin}$ in a straightforward manner. On the classical side, it can be shown

that the constraint $D(f)$ is the infinitesimal generator of the one-parameter family of diffeomorphisms defined by the vector field f. In particular, a physical state is one that is invariant under diffeomorphisms, which simply reflects the invariance of GR under passive (spatial) diffeomorphisms.

On the quantum side, however, it is straightforward to implement the action of piecewise analytic diffeomorphisms on $\mathcal{H}_{kin}$: Remember that one can think of $\mathcal{H}_{kin}$ as consisting of functions $f : \mathcal{A} \to \mathbb{C}$, which are cylindrical over some graph γ. The space of quantum configurations $\mathcal{A}$, i.e. the space of (distributional) connections on Σ carries a natural action of the diffeomorphism group Diff Σ. An element $\phi \in$ Diff Σ simply acts by $A \to \phi^* A$ on a (distributional) connection A. With this, one can simply define the action of Diff Σ on $\mathcal{H}_{kin}$ by

$$\alpha_\phi f(A) \;:=\; f(\phi^* A),$$

where $\phi^* A$ is the pullback of the connection A under the diffeomorphism ϕ. Note that this definition maps

$$\alpha_\phi \, \mathcal{H}_\gamma \;\longrightarrow\; \mathcal{H}_{\phi(\gamma)}. \tag{2.9}$$

Here $\phi(\gamma)$ is the image of γ under ϕ. This shows that one cannot take arbitrary smooth ϕ, but has to restrict to analytic diffeomorphisms, since these map a graph consisting of analytic edges into one consisting again of analytic edges.

Note that the action (2.9) is not weakly continuous in ϕ, since two graphs can be arbitrary "close" to each other, but still not intersecting, which means that their corresponding Hilbert spaces are mutually orthogonal subspaces of $\mathcal{H}_{kin}$. This fits nicely into the picture, since the notion of "being close to each other" only has a meaning on manifolds with metric, and LQG is a quantum theory on a topological manifold only, since the metric itself is a dynamical object, and not something given from the outset.

The Hamiltonian constraints $H(n)$ could in fact be promoted to operators $\hat{H}(n)$ on $\mathcal{H}_{kin}$ [25]. But, the solution of this constraint, i.e. determining the set of (generalized) vectors satisfying $\hat{H}(n)\psi_{phys} = 0$ is still elusive. Also, since these operators exhibit a highly nontrivial bracket structure, it is not clear whether they resemble their classical counterpart (2.1). Moreover, these operators cannot be defined on the diffeomorphism-invariant Hilbert space $\mathcal{H}_{\mathrm{diff}}$. To remedy these issues, a modification to the algebra (2.1) has been proposed, the so-called master constraint programme. By replacing all $\hat{H}(n)$ by one operator $\hat{M}$, one can solve the above issues [26, 27]. Still, the solution of this constraint is quite nontrivial, although some steps into this direction have been undertaken [20].

3 Complexifier coherent states

An important question in LQG is whether the theory contains classical GR in some sort of semiclassical limit [1, 17, 20]. The transition from quantum to classical

behavior in the case of, say, a quantum mechanical particle moving in one dimension can be seen best with the help of the harmonic oscillator coherent states (HOSZ)

$$|z\rangle \;=\; \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\,|n\rangle. \tag{3.1}$$

They can be seen as minimal uncertainty states, or states that correspond to the system of being in a quantum state close to a classical phase space point. With these states, one can not only investigate the transition from quantum to classical behavior of a system, but one can also try to say something about the dynamics of the quantum system by considering solutions to the classical equations of motion.

This has led people to consider, whether states with equally pleasant properties also exist for LQG. In [17], states in $\mathcal{H}_{kin}$ have been proposed that have been constructed by the so-called complexifier method, first brought up in [13, 14]. They have been investigated in [18, 19], and the properties of these states seem to make them ideally suited for the semiclassical analysis of the kinematical sector of LQG [20].

The complexifier coherent states are defined for each graph $\gamma \subset \Sigma$ separately, and each of these Hilbert spaces is, by (2.3), a tensor product of $L^2(SU(2), d\mu_H)$-spaces. Also the complexifier coherent states on $\mathcal{H}_\gamma$ are defined as a tensor product of complexifier coherent states on $L^2(SU(2), d\mu_H)$. In fact, the complexifier procedure is quite general and works for every compact Lie group G, and is able to define a state on $L^2(G, d\mu_H)$. This comes in handy, since Yang-Mills field theory coupled to gravity can be treated at the kinematical level, simply by replacing $SU(2)$ by a compact gauge group G in the whole construction. There are in fact arguments that, in the semiclassical limit, the qualitative behavior of calculations in LQG will not change if one replaces $SU(2)$ by $U(1)^3$. This replacement has been used widely during the investigation of the semiclassical limit of LQG [20]. The fact that $U(1)^3$ is abelian is a tremendous simplification to the calculations.

Thus, in the following we will give the definition of the complexifier coherent states for arbitrary gauge groups, where the cases of $G = U(1)$, $U(1)^3$ and $SU(2)$ are of ultimate interest for the geometry degrees of freedom of LQG.

3.1 General gauge groups

Consider quantum mechanics on a compact Lie group G, which is associated to the Hilbert space $L^2(G,\, d\mu_H)$, where $d\mu_H$ is the normalized Haar measure on G. The classical configuration space is G, and the corresponding phase space is

$$T^*G \;\simeq\; G \times \mathbb{R}^{\dim G} \;\simeq\; G^{\mathbb{C}}. \tag{3.2}$$

Here, $G^{\mathbb{C}}$ is the complexification of G, generated by the complexification of the Lie algebra of G, $\mathfrak{g} \otimes \mathbb{C}$. The complexifier coherent states are then defined by

$$\psi_g^t(h) \;:=\; \left(e^{\Delta \frac{t}{2}}\,\delta_{h'}(h)\right)\Big|_{h' \to g}. \tag{3.3}$$

The $\delta_{h'}(h)$ is the delta distribution on G with respect to $d\mu_H$, centered around $h' \in G$, Δ is the Laplacian operator and $h' \to z$ is the analytic continuation from

$h' \in G$ to $g \in G^{\mathbb{C}}$. The fact that the spectrum of Δ grows quadratically for large eigenvalues makes sure that the expression in the brackets is in fact a smooth function on G, thus ensuring that $\psi_g^t \in L^2(G,\, d\mu_H)$.

These states are named complexifier coherent states, since, instead of $-\Delta$, one could have taken any quantization of a phase space function C (with spectrum bounded from below and spectrum growing at least as $l^{1+\epsilon}$, in order for the above expression to make sense). The function C is called a complexifier, since it provides an explicit diffeomorphism between $T^*(G) \simeq G^{\mathbb{C}}$, such that the element $g \in G^{\mathbb{C}}$ actually carries a physical interpretation as a point in phase space. This diffeomorphism is, for the complexifier $\hat{C} = -\Delta$, given by

$$T^*G \;\simeq\; G \times \mathbb{R}^{\dim G} \;\ni\; (h, \vec{p}) \;\longmapsto\; \exp\left(-i\frac{\tau_I}{2}p^I\right) h \;\in\; G^{\mathbb{C}}$$

which is the inverse of the polar decomposition of elements in $G^{\mathbb{C}}$, while the τ_I are basis elements of $\mathfrak{g}$. A priori, which complexifier $\hat{C}$ one chooses is not fixed. In the context of LQG, one can, given a graph γ, choose a classical function C adapted to this graph, such that its quantization $\hat{C}$ is - restricted to $\mathcal{H}_\gamma$ - just the Laplacian $-\Delta$ on each edge. See [28] for details and a discussion of this operator.

From (3.3) one can deduce a more tractable form of the complexifier coherent states given by

$$\psi_g^t(h) \;=\; \sum_\pi e^{-l_\pi} d_\pi \;\; \mathrm{tr}\; \pi(gh^{-1}) \tag{3.4}$$

where the sum runs over all irreducible finite-dimensional representations π of G. In the specific case of $G = U(1)$ and $G = SU(2)$, the states (3.4) have been investigated [17, 18, 19], and their properties are known quite well. In particular, they approximate the quantum operators up to small fluctuations, the width of which is proportional $\sqrt{t}$, which identifies t as the parameter measuring the semiclassicality scale. For kinematical states in LQG being close to some smooth space-time, at the scale of say the LHC t is of the order of $l_p^2/(10^{-18}\text{ cm})^2$, i.e. about 10^{-30}!

The states (3.4) are complexifier coherent states for quantum mechanics on G. Technically, this is equivalent to a graph consisting of one edge. For graphs γ being built of many edges $e_1, \ldots e_E$, one can, since $L^2(G, d\mu_H)^{\otimes E} \;=\; L^2(G^E, d\mu_H^{\otimes E})$, simply construct a state by taking the tensor product over all edges:

$$\psi^t_{g_1,\ldots,g_E}(h_1, \ldots, h_E) \;=\; \prod_{m=1}^{E} \psi^t_{g_m}(h_m). \tag{3.5}$$

Note that this tensor product contains no information about which edges are connected to each other and which are not.

The complexifier coherent states on a graph are labeled by elements $g_m \in G^{\mathbb{C}}$. In particular, for the cases of interest for LQG, these spaces are

$$U(1)^{\mathbb{C}} \simeq \mathbb{C}\backslash\{0\}$$
$$SU(2)^{\mathbb{C}} \simeq SL(2, \mathbb{C}).$$

As already stated, the complexified groups $G^{\mathbb{C}}$ are diffeomorphic to the tangent bundle of the groups T^*G themselves. So, the complexifier coherent states are labeled by elements of the classical phase space. A state labeled by $g_1, \ldots, g_E$ corresponds to a state being close to the classical phase space point corresponding to $g_1, \ldots, g_E$. This interpretation is supported by the fact that - as could be shown for the cases $G = U(1)$ and $G = SU(2)$ - the expectation values of quantizations of holonomies and fluxes coincide - up to orders of $\hbar$ - with the classical holonomies and fluxes determined by the phase space point corresponding to $g_1, \ldots, g_E$ [19]. Furthermore, the overlap between two complexifier coherent states is sharply peaked [19]:

$$\frac{\left|\left\langle \psi^t_{g_1,\ldots,g_E} \middle| \psi^t_{h_1,\ldots,h_E} \right\rangle\right|^2}{\left\|\psi^t_{g_1,\ldots,g_E}\right\|^2 \left\|\psi^t_{h_1,\ldots,h_E}\right\|^2} \;=\; \left\{ \begin{array}{cc} 1 & g_m = h_m \text{ for all } m \\ \begin{array}{c}\text{decaying exponentially} \\ \text{as } t \to 0\end{array} & \text{else} \end{array} \right.$$

This shows that the complexifier coherent states (3.4) are suitable to approximate the kinematical operators of LQG quite well. Although the original LQG has been constructed with $G = SU(2)$, it has been shown that in the semiclassical regime, the group $SU(2)$ can be replaced by $U(1)^3$ without changing the qualitative behavior of expectation values or fluctuations. On the other hand, with this trick calculations simplify tremendously, since $U(1)^3$ is an abelian group. Furthermore, $U(1)^3$ is simply the Cartesian product of three copies of $U(1)$, which also completely determines the set of irreducible representations of $U(1)^3$, such that a complexifier coherent state on $U(1)^3$ is nothing but a product of three states on $U(1)$:

$$\psi^t_{(g_1,g_2,g_3)}(h_1, h_2, h_3) \;=\; \psi^t_{g_1}(h_1)\,\psi^t_{g_2}(h_2)\,\psi^t_{g_3}(h_3).$$

This is, of course, true for any Cartesian product between - not necessarily distinct - compact Lie groups.

Since the properties of complexifier coherent states on $U(1)^3$ can be investigated by considering states on $U(1)$, we will work with the latter from now on.

3.2 The case of $G = U(1)$

In the last section, the general definition of complexifier coherent states for arbitrary compact Lie groups G has been given. In this section, we will shortly review these states for the simplest case of $G = U(1)$, since we will work with these states in the rest of the article.

From (3.4), we can immediately deduce the explicit form of the complexifier coherent states, since all irreducible representations of $U(1)$ are known and one-dimensional:

$$\psi^t_z(\phi) \;=\; \sum_{n\in\mathbb{Z}} e^{-n^2 \frac{t}{2}}\, e^{-in(z-\phi)} \tag{3.6}$$

for $g = e^{iz}$ and $h = e^{i\phi}$. With the Poisson summation formula, this expression can be rewritten as

$$\psi^t_g(h) \;=\; \sqrt{\frac{2\pi}{t}} \sum_{n\in\mathbb{Z}} e^{-\frac{(z\,-\,\phi\,-\,2\pi n)^2}{2t}}. \tag{3.7}$$

The inner product of two of these states is then

$$\langle \psi_g^t | \psi_{g'}^t \rangle \;=\; \sqrt{\frac{\pi}{t}} \sum_{n\in\mathbb{Z}} e^{-\frac{(\bar{z}\,-\,z'\,-\,2\pi n)^2}{t}}. \tag{3.8}$$

There is a way to interpret (3.8) geometrically. This makes use of the fact that $G^{\mathbb{C}} = \mathbb{C}\backslash\{0\}$ comes with a pseudo-Riemannian metric provided by the Killing form on its Lie algebra. On arbitrary Lie groups G, this metric is denoted, in components, by

$$h_{IJ} \;=\; -\frac{1}{\dim G}\ \mathrm{tr}\ \left(g^{-1}\partial_I g\, g^{-1}\partial_J g\right). \tag{3.9}$$

Choosing the chart $z \to e^{iz}$ on $\mathbb{C}\backslash\{0\}$, the metric (3.9) simply takes the form $h = 1$. Note that the geodesics through $1 \in \mathbb{C}\backslash\{0\}$ with respect to this metric are given by

$$t \;\longmapsto\; e^{itz} \tag{3.10}$$

for some $z \in \mathbb{C}$, which corresponds to the velocity of the geodesic at $t = 0$. Note also that geodesics can be transported via group multiplication, since the metric is defined via group translation. In particular, if $\gamma(t)$ is a geodesic on $\mathbb{C}\backslash\{0\}$, then $g\gamma(t)$ is also one for any $g \in \mathbb{C}\backslash\{0\}$.

With h one can define the complex length-square of a geodesic, or any other regular curve γ on $\mathbb{C}\backslash\{0\}$, via

$$l^2(\gamma) \;:=\; \left(\int dt\sqrt{h(\gamma(t))\dot{\gamma}(t)\dot{\gamma}(t)}\right)^2. \tag{3.11}$$

Note that this gives a well-defined complex number, since the square of a complex number is defined up to a sign, and this sign is chosen continuously on the whole curve, which gives a unique choice since the curve is regular, i.e. its velocity vector vanishes nowhere. So, the integral is determined up to a sign, the square of which is then well-defined.

Let $g,\ h \in \mathbb{C}\backslash\{0\}$, and $\gamma : [0,1] \to \mathbb{C}\backslash\{0\}$ be a geodesic from g to h. It is straightforward to compute that such a geodesic is not unique, but, for $g = e^{iw}$ and $h = e^{iz}$ (where z and w are determined up to $2\pi n$ for some $n \in \mathbb{Z}$), is given by

$$\gamma(t) \;=\; e^{iw}\, e^{it(z\,-\,w\,-\,2\pi n)}. \tag{3.12}$$

for any $n \in \mathbb{Z}$. By changing n, one ranges through the set of geodesics from g to h. The complex length square of the (3.12) can easily be computed to be

$$l(\gamma)^2 \;=\; (z \,-\, w \,-\, 2\pi n)^2. \tag{3.13}$$

This shows that one can write the inner product between two complexifier coherent states as sum over complex lengths of geodesics:

$$\langle \psi_g^t | \psi_h^t \rangle \;=\; \sum_{\substack{\gamma \text{ geodesic} \\ \text{from } g^c \text{ to } h}} e^{-\frac{l(\gamma)^2}{t}} \tag{3.14}$$

with $g^c := \bar{g}^{-1}$.

Although this seems to be too much effort to rewrite a simple expression like (3.8), we will encounter a similar expression in [24] for the case of $SU(2)$-complexifier coherent states. This relates the complexifier coherent states with the geometry of the corresponding group, which is given by the Killing metric (3.9). We will comment on this at the end of [24].

4 Gauge-invariant coherent states with gauge group $G = U(1)$

4.1 The gauge-invariant sector

In the following, we will describe the Hilbert space invariant under the Gauss gauge transformation group. Since this gauge transformation group $\mathcal{G}$ leaves every graph invariant, we can restrict ourselves to the case of one graph, in particular

$$\mathcal{P} \varinjlim H_\gamma \;=\; \varinjlim \mathcal{P}\mathcal{H}_\gamma.$$

So we can consider the gauge-invariant cylindrical functions on each graph separately.

The gauge-invariant cylindrical functions on a graph γ with E edges and V vertices can be described in terms of singular cohomology classes with values in the gauge group. In particular, every Hilbert space $\mathcal{H}_\gamma$ is canonically isomorphic to an L^2-space:

$$H_\gamma \;\simeq\; L^2\left(G^E,\, d\mu_H^{\otimes E}\right), \tag{4.1}$$

where $d\mu_H$ is the normalized Haar measure on the compact Lie group G. It is known that the gauge-invariant Hilbert space is then canonically isomorphic to an L^2-space over the first simplicial cohomology group of γ with values in the gauge group G:

$$\mathcal{P}H_\gamma \;\simeq\; L^2\left(H^1(\gamma, G),\, d\mu\right), \tag{4.2}$$

with a certain measure $d\mu$. For abelian gauge groups G, first cohomology group of γ with values in G is given by

$$H^1(\gamma, G) \;\simeq\; G^{E-V+1}, \tag{4.3}$$

and $d\mu = d\mu_H^{\otimes E-V+1}$ is the $E-V+1$-fold tensor product of the Haar measure on G. See appendix A for a summary of abelian cohomology groups on graphs and their relation to gauge-invariant functions. For non-abelian gauge groups G a similar result holds, while the definition of the first cohomology class requires more care. This case will be dealt with in [24], and we stay with abelian G in this article.

4.2 Gauge-invariant coherent states

We now come to the main part of this article: The computation of the gauge-invariant coherent states. We will derive a closed form for them, revealing the intimate relationship between the gauge-invariant degrees of freedom and the graph topology. From the explicit form we will be able to compute the overlap between two gauge-invariant coherent states, which will allow for an interpretation as semi-classical states for the gauge-invariant sector of the theory.

The gauge-invariant coherent states are obtained by applying the gauge projector (2.8) to the complexifier coherent states on a graph (3.5), (3.6), i.e.

$$\Psi^t_{[g_1,\ldots,g_E]}([h_1,\ldots h_E]) \;=\; \mathcal{P}\psi^t_{g_1,\ldots,g_E}(h_1,\ldots,h_E). \tag{4.4}$$

It is known that the set of gauge-invariant functions can be described in terms of functions on the first cohomology class of the graph. See the appendix for details. In particular, if the graph has E edges and V vertices, i.e. the gauge-variant configuration space is diffeomorphic to $U(1)^E$, then the gauge-invariant configuration space is diffeomorphic to $U(1)^{E-V+1}$. This might raise the hope that these states somehow resemble complexifier coherent states on the gauge-invariant configuration space $U(1)^{E-V+1}$. We will see that this is not quite true, but near enough.

The fact that the gauge group is abelean is a great simplification: It allows us to pull back all group multiplications to simple addition on the algebra, simply due to the fact that $\exp\, iz\, \exp\, iw = \exp\, i(z+w)$. This will allow us to explicitly perform the gauge integrals for arbitrary graphs, and obtain a formula for the gauge-invariant coherent states that only depends on gauge-invariant combinations of $h_k = \exp\, i\phi_k$ and $g_k = \exp\, iz_k$, as well as topological information about the graph, in particular its incidence matrix.

4.3 Basic graph theory

In order to be able to deal with the expressions for all graphs, we start with some basics of graph theory. All the material, as well as all the proofs, can be found in [29] and the references therein.

Definition 4.1 *Let γ be a directed graph with V vertices and E edges. Let the edges be labeled by numbers $1,\ldots,E$ and the vertices by numbers $1,\ldots,V$. Then the* incidence matrix $l \in \mathrm{Mat}(E\times V,\mathbb{Z})$ *is defined by the following rule:*

$$\begin{aligned}
l_{kl} &:= 1 && \text{if the edge } k \text{ ends at vertex } l\\
l_{kl} &:= -1 && \text{if the edge } k \text{ starts at vertex } l\\
l_{kl} &:= 0 && \text{else.}
\end{aligned}$$

Note in particular that, if edge k starts *and* ends at vertex l, i.e. the edge k is a loop, then $l_{kl} = 0$ as well. Since either an edge is a loop or starts at one and ends

at some other vertex, every line of the matrix l is either empty, or contains exactly one 1 and one -1. With the definition

$$u \;:=\; \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \;\in\; \mathbb{R}^V, \tag{4.5}$$

we immediately conclude

$$l^T u \;=\; 0. \tag{4.6}$$

Definition 4.2 *Let γ' be a graph. If γ' contains no loops, then γ' is said to be a* tree*. If $\gamma' \subset \gamma$ is a subgraph, then γ' is said to be a* tree in γ*. If $\gamma' \subset \gamma$ is a subgraph that meets every vertex of γ, then γ' is said to be a* maximal tree (in γ)*.*

Lemma 4.1 *Every graph γ has a maximal tree as subgraph. Every tree has $V = E - 1$ vertices.*

Maximal trees in graphs are not unique. It is quite easy to show that every function cylindrical over a graph γ is gauge equivalent to a function cylindrical over γ, which is constant on the edges corresponding to a maximal tree. This will be used later, and by the preceding Lemma we immediately conclude that the number of gauge-invariant degrees of freedoms on a graph with V vertices and E edges is $E - V + 1$ for Abelian gauge theories. This will be seen explicitly at the end of this section.

The following theorem relates the numbers of different possible maximal trees to the incidence matrix.

Theorem 4.1 *(Kirchhoff) Let γ be a graph and l its incidence matrix. Then the* Kirchoff-matrix $K := ll^T$ *has nonnegative eigenvalues*

$$0 \;=\; \mu_1 \leq \mu_2 \leq \cdots \leq \mu_V.$$

The lowest eigenvalue is $\mu_1 = 0$, and the degeneracy of 0 *is the number of connected components of the graph γ. Furthermore, the product of all nonzero eigenvalues*

$$G \;:=\; \frac{1}{V} \prod_{\mu_k \neq 0} \mu_k$$

is the number of different maximal trees in γ.

With this machinery, we will be able to perform the gauge integral for arbitrary graphs. This will include some kind of gauge-fixing procedure, which will make use of a maximal tree.

4.4 Gauge-variant coherent states and the gauge integral

The Abelian nature of the gauge group allows us to pull back the group multiplication to addition on the Lie algebra. This is why throughout this chapter we will, instead of elements $h \in U(1)$, deal with $\phi \in \mathbb{R}$ by $h = \exp i\phi$, and instead of elements $g \in \mathbb{C}\backslash\{0\}$, we will work with the corresponding $z \in \mathbb{C}$ such that $g = \exp iz$, always having in mind that ϕ and z are only defined modulo $2\pi n$ for $n \in \mathbb{Z}$.

We will denote vectors (of any length) as simple letters $z, \phi, \tilde{\phi}, m, \ldots$ and their various components with indices: $z_k, \phi_k, \tilde{\phi}_k, \ldots$. The particular range of the indices will be clear from the context, but we will still repeat it occasionally.

The gauge-variant coherent states on a graph γ with E edges are simply given by the product

$$\psi_z^t(\phi) \;=\; \prod_{k=1}^{E} \sum_{m_k \in \mathbb{Z}} e^{-m_k^2 \frac{t}{2}}\, e^{i m_k (z_k - \phi_k)} \tag{4.7}$$

where $z_k = \phi_k - ip_k,\; k = 1, \ldots, E$ is labeling the points in phase space where the coherent states are peaked. With the Poisson summation formula one can rewrite this as

$$\psi_z^t(\phi) \;=\; \sqrt{\frac{2\pi}{t}}^{E} \sum_{m_1, \ldots, m_E \in \mathbb{Z}} \exp\left(-\sum_{k=1}^{E} \frac{(z_k - \phi_k - 2\pi m_k)^2}{2t}\right) \tag{4.8}$$

We will now perform the gauge integral

$$\begin{aligned}\Psi_{[z]}^t(\phi) \;&=\; \int_G d\mu_H(\tilde{\phi})\; \psi_{\alpha_{\tilde{\phi}} z}(\phi) &(4.9)\\ &=\; \sqrt{\frac{2\pi}{t}}^{E} \int_{[0,2\pi]^V} \frac{d\tilde{\phi}_1}{2\pi} \cdot \ldots \cdot \frac{d\tilde{\phi}_V}{2\pi} \sum_{m_1, \ldots, m_E \in \mathbb{Z}} \exp\left(-\sum_{k=1}^{E} \frac{(A_k + l_{ka}\tilde{\phi}_a - 2\pi m_k)^2}{2t}\right)\end{aligned}$$

with $A = z - \phi$, and where l_{ka} are the components of the transpose l^T of the incidence matrix.

In what follows, we will use the symmetries of this expression, together with a gauge-fixing procedure, to separate the gauge degrees of freedom from the gauge-invariant ones. The integrals will then be performable analytically, and the resulting expression can then be interpreted as states being peaked on gauge-invariant quantities.

To simplify the notation, we will assume, without loss of generality, that γ is connected. Furthermore choose, once and for all, a maximal tree $\tau \subset \gamma$. Choose the numeration of vertices and edges of γ according to the following scheme:

Start with the maximal tree τ. The tree consists of V vertices and $V - 1$ edges. Call a vertex that has only one outgoing edge (in τ, not necessarily in γ) an outer end of τ. Remove one outer end and the corresponding edge from τ and obtain a

smaller subgraph $\tau^1 \subset \gamma$, which is also a tree. Label the removed vertex with the number 1, and do so with the removed edge as well. So this gives you v_1 and e_1. From τ^1, remove an outer end and the corresponding edge, and label them v_2 and e_2, and obtain a yet smaller tree $\tau^2 \subset \tau^1 \subset \tau \subset \gamma$. Repeat this process until τ has been reduced to $\tau^{(V-1)}$, which is a point. This way, one has obtained $v_1, \ldots, v_{V-1}$ and $e_1, \ldots, e_{E-1}$. Call the last, remaining vertex v_V. Label the edges that do not belong to τ by $e_V, e_{V+1}, \ldots, e_E$ in any order.

Choosing the numeration of the vertices and the edges in the above manner will help us in rewriting the expression (4.9). First we note that the first $V-1$ edges and the first V vertices constitute the tree, the last $E-V+1$ edges constitute what is not the tree in γ. Furthermore, with this numeration, the edge e_k is starting or ending at vertex v_k for $k = 1, \ldots, V-1$. In particular, the diagonal elements of the incidence matrix are all (except maybe the last one) nonzero: $l_{kk} \neq 0$ for $k = 1, \ldots, V-1$.

Definition 4.3 *Let γ be a graph, with vertices $v_1, \ldots v_V$ and edges $e_1, \ldots, e_E$. Between two vertices v_k and v_l there is a unique path in τ, since a tree contains no loops. Call v_k being* before *v_l, if this path includes e_k, otherwise call v_k being* after *v_l.*

Note that a vertex cannot be both before and after another vertex, but two vertices can both be before or both be after each other.

The numeration we have chosen has the following consequence: For each vertex v_k one has that for all v_l such that v_k is after v_l, that $l \leq k$. The converse need not be true. Note further that every vertex is before itself, by this definition. Also, since e_V is not an edge of the graph, it does not even have to be touching v_V. So, the question of whether v_V is before or after any other vertex makes no sense in this definition (But note that it does make sense to ask whether any vertex is before or after v_V).

We now rewrite formula (4.9), by replacing the integrals over $[0, 2\pi]$ by integrals over $\mathbb{R}$. We do this inductively over the vertices from v_1 to v_{V-1}. Consider the E terms constituting the sum in the exponent in

$$\begin{aligned}\Psi^t_{[z]}(\phi) &= \int_G d\mu_H(\tilde{\phi})\; \psi_{\alpha_{\tilde{\phi}} z}(\phi) \\ &= \sqrt{\frac{2\pi}{t}}^E \int_{[0,2\pi]^V} \frac{d\tilde{\phi}_1}{2\pi} \cdot \ldots \cdot \frac{d\tilde{\phi}_V}{2\pi} \sum_{m_1,\ldots,m_E \in \mathbb{Z}} \exp\left(- \sum_{k=1}^{E} \frac{(A_k + l_{ka}\tilde{\phi}_a - 2\pi m_k)^2}{2t} \right).\end{aligned}$$

In some of them $\tilde{\phi}_1$ appears, in some of them it does not, precisely if either $l_{k1} \neq 0$ or $l_{k1} = 0$. Note that $\tilde{\phi}_1$ definitely appears in the first term, by the above considerations. If $\tilde{\phi}_1$ appears in the k-th term other than $k = 1$, shift the infinite sum over m_k by $m_k + l_{11} l_{k1} m_1$. The result of this is that, since $l_{k1}^2 = l_{11}^2 = 1$ for these k, after this shift $\tilde{\phi}_1$ appears always in the combination $l_{11}\tilde{\phi}_1 - 2\pi m_1$ in all the factors. Now we can employ the formula

$$\int_{[0,2\pi]} \frac{d\tilde{\phi}}{2\pi} \sum_{m \in \mathbb{Z}} f(\tilde{\phi} \pm 2\pi m) \;=\; \frac{1}{2\pi} \int_{\mathbb{R}} d\tilde{\phi}\; f(\tilde{\phi}) \tag{4.10}$$

and, regardless of whether $l_{11} = +1$ or $l_{11} = -1$, have

$$(4.9) = \sqrt{\frac{2\pi}{t}}^E \int_{\mathbb{R}} \frac{d\tilde{\phi}_1}{2\pi} \int_{[0,2\pi]^{V-1}} \frac{d\tilde{\phi}_2}{2\pi} \cdot \ldots \cdot \frac{d\tilde{\phi}_V}{2\pi}$$

$$\times \sum_{m_2,\ldots,m_E\in\mathbb{Z}} \exp\left(-\frac{(A_1 + l_{1a}\tilde{\phi}_a)^2}{2t} - \sum_{k=2}^{E} \frac{(A_k + l_{ka}\tilde{\phi}_a - 2\pi m_k)^2}{2t}\right).$$

This being the beginning of the induction, we now describe the induction step from l to $l+1$ by the following technical lemma. By this we will be able to extend all integration ranges over all of $\mathbb{R}$, instead of finite intervals, which will turn out to be very useful.

Lemma 4.2 *Let γ be a graph with V vertices, E edges, and l be its incidence matrix. Let $A \in \mathbb{C}^E$ and $t > 0$, then we have, for $1 \le l \le V-1$:*

$$\sqrt{\frac{2\pi}{t}}^E \int_{\mathbb{R}^{l-1}} \frac{d\tilde{\phi}_1}{2\pi} \cdot \ldots \cdot \frac{d\tilde{\phi}_{l-1}}{2\pi} \int_{[0,2\pi]^{V-l+1}} \frac{d\tilde{\phi}_l}{2\pi} \cdot \ldots \cdot \frac{d\tilde{\phi}_V}{2\pi}$$

$$\times \sum_{m_l,\ldots,m_E\in\mathbb{Z}} \exp\left(-\sum_{k=1}^{l-1} \frac{(A_k + l_{ka}\tilde{\phi}_a)^2}{2t} - \sum_{k=l}^{E} \frac{(A_k + l_{ka}\tilde{\phi}_a - 2\pi m_k)^2}{2t}\right)$$

$$= \sqrt{\frac{2\pi}{t}}^E \int_{\mathbb{R}^{l}} \frac{d\tilde{\phi}_1}{2\pi} \cdot \ldots \cdot \frac{d\tilde{\phi}_{l}}{2\pi} \int_{[0,2\pi]^{V-l}} \frac{d\tilde{\phi}_{l+1}}{2\pi} \cdot \ldots \cdot \frac{d\tilde{\phi}_V}{2\pi}$$

$$\times \sum_{m_{l+1},\ldots,m_E\in\mathbb{Z}} \exp\left(-\sum_{k=1}^{l} \frac{(A_k + l_{ka}\tilde{\phi}_a)^2}{2t} - \sum_{k=l+1}^{E} \frac{(A_k + l_{ka}\tilde{\phi}_a - 2\pi m_k)^2}{2t}\right).$$

Proof: Note that we just proved the formula for $l = 1$. In the proof for arbitrary $1 \le l \le V-1$ we will use the notion of vertices being before and after one another.

Consider all vertices v_k being after v_l, other than v_l itself. By construction, for all such k, we have $k < l$, so by induction hypothesis, the integration over all these v_k runs over all of $\mathbb{R}$, not over just the interval $[0, 2\pi]$ any more. Consequently, the sum over these m_k is not appearing any longer. So we can shift the integration range by $+2\pi l_{ll} m_l$.

This will affect the terms in the first sum in

$$\exp\left(-\sum_{k=1}^{l-1} \frac{(A_k + l_{ka}\tilde{\phi}_a)^2}{2t} - \sum_{k=l}^{E} \frac{(A_k + l_{ka}\tilde{\phi}_a - 2\pi m_k)^2}{2t}\right) \qquad (4.11)$$

in the following way: Let $k < l$. The edge e_k then belongs to the tree τ, and thus v_l is either after both vertices e_k touches, or before both vertices. If v_l is before both, the term does not change at all, since the two $\tilde{\phi}_a$ in it are not shifted. If v_l is after both and is not itself one of the two vertices, then the term gets changed by

$$(A_k + l_{ka}\tilde{\phi}_a)^2 \;\longrightarrow\; (A_k + l_{ka}\tilde{\phi}_a \;\pm 2\pi l_{ll} m_l \;\mp 2\pi l_{ll} m_l)^2 \;=\; (A_k + l_{ka}\tilde{\phi}_a)^2,$$

since the two $\tilde{\phi}_a$ in a term always appear with opposite sign. So these terms do not change, too. If v_l is after both vertices that touch e_k and is itself one of it (i.e. e_k is an edge adjacent to e_l, linked by v_l), then the corresponding term changes by

$$
\begin{aligned}
(A_k + l_{ka}\tilde{\phi}_a)^2 &= (A_k + l_{kl}\tilde{\phi}_l + l_{kk}\tilde{\phi}_k)^2 \;=\; (A_k + l_{kk}(\tilde{\phi}_k - \tilde{\phi}_l))^2\\
&\longrightarrow\; (A_k + l_{kk}(\tilde{\phi}_k - \tilde{\phi}_l + 2\pi l_{ll} m_l))^2,
\end{aligned}
$$

where $l_{kl} = -l_{ll}$ and $l_{ll}^2 = 1$ have been used.

So, after this shift, in all terms in the first sum in (4.11) $\tilde{\phi}_l$ has been replaced by $\tilde{\phi}_l - 2\pi l_{ll} m_l$. The first term of the second sum reads

$$
(A_l + l_{ll}(\tilde{\phi}_l - \tilde{\phi}_a) - 2\pi m_l)^2 \;=\; (A_l - l_{ll}\tilde{\phi}_a + l_{ll}(\tilde{\phi}_l - 2\pi l_{ll} m_l))^2,
$$

where v_a is the other vertex touching e_l, apart from v_l. So also in this term $\tilde{\phi}_l$ and m_l appear in the combination $\tilde{\phi}_l - 2\pi l_{ll} m_l$.

The terms $k = l+1$ till $k = E - V + 1$ remain unchanged, since they all correspond to edges that lie between vertices v_a such that v_l is before both v_a, and these $\tilde{\phi}_a$ are hence not shifted.

The terms $k = E - V + 2$ till $k = E$ in (4.11), on the other hand, correspond to edges that lie between two vertices such that v_l could be before the one and after the other. This is due to the fact that these edges do not belong to the maximal tree T any longer. So in these terms, a shift by $\pm 2\pi l_{ll} m_l$ could have occurred by the shift of integration range. But in all these terms, there is still a term $-2\pi m_k$ present, and the sum over these m_k is still performed. So, by appropriate shift of these summations, similar to the ones performed in the induction start, one can subsequently produce or erase terms of the form $\pm 2\pi l_{ll} m_l$ in all of the terms corresponding to $k = E - V + 2$ till $k = E$. Since there are enough summations left, one has enough freedom to produce a $\pm 2\pi l_{ll} m_l$, where $\tilde{\phi}_l$ is present, or erase all terms with m_l, where $\tilde{\phi}_l$ is not present.

Thus, in the end, we again have a function only depending on $\tilde{\phi}_l - 2\pi l_{ll} m_l$, and thus we can again apply formula (4.10), and, regardless of the sign of l_{ll}, erase the infinite sum over μ_l, obtaining an integration range of $\tilde{\phi}_l$ over all of $\mathbb{R}$:

$$
\begin{aligned}
&\sqrt{\frac{2\pi}{t}}^{E} \int_{\mathbb{R}^{l-1}} \frac{d\tilde{\phi}_1}{2\pi} \cdot \ldots \cdot \frac{d\tilde{\phi}_{l-1}}{2\pi} \int_{[0,2\pi]^{V-l+1}} \frac{d\tilde{\phi}_l}{2\pi} \cdot \ldots \cdot \frac{d\tilde{\phi}_V}{2\pi}\\
&\qquad\qquad \times \sum_{m_l,\ldots,m_E \in \mathbb{Z}} \exp\left(-\sum_{k=1}^{l-1} \frac{(A_k + l_{ka}\tilde{\phi}_a)^2}{2t} - \sum_{k=l}^{E} \frac{(A_k + l_{ka}\tilde{\phi}_a - 2\pi m_k)^2}{2t} \right)\\
=&\sqrt{\frac{2\pi}{t}}^{E} \int_{\mathbb{R}^{l}} \frac{d\tilde{\phi}_1}{2\pi} \cdot \ldots \cdot \frac{d\tilde{\phi}_{l}}{2\pi} \int_{[0,2\pi]^{V-l}} \frac{d\tilde{\phi}_{l+1}}{2\pi} \cdot \ldots \cdot \frac{d\tilde{\phi}_V}{2\pi}\\
&\qquad\qquad \times \sum_{m_{l+1},\ldots,m_E \in \mathbb{Z}} \exp\left(-\sum_{k=1}^{l} \frac{(A_k + l_{ka}\tilde{\phi}_a)^2}{2t} - \sum_{k=l+1}^{E} \frac{(A_k + l_{ka}\tilde{\phi}_a - 2\pi m_k)^2}{2t} \right).
\end{aligned}
$$

This was the claim of the Lemma.

An immediate corollary of Lemma 4.2 is that

$$
\begin{aligned}
&\sqrt{\frac{2\pi}{t}}^E \int_{[0,2\pi]^V} \frac{d\tilde{\phi}_1}{2\pi} \cdot \ldots \cdot \frac{d\tilde{\phi}_V}{2\pi} \sum_{m_1,\ldots,m_E\in\mathbb{Z}} \exp\left(-\sum_{k=1}^{E} \frac{(A_k + l_{ka}\tilde{\phi}_a - 2\pi m_k)^2}{2t}\right) \\
&= \sqrt{\frac{2\pi}{t}}^E \int_{\mathbb{R}^{V-1}} \frac{d\tilde{\phi}_1}{2\pi} \cdot \ldots \cdot \frac{d\tilde{\phi}_{V-1}}{2\pi} \int_0^{2\pi} \frac{d\tilde{\phi}_V}{2\pi} \qquad (4.12)\\
&\qquad \times \sum_{m_V,\ldots,m_E\in\mathbb{Z}} \exp\left(-\sum_{k=1}^{V-1} \frac{(A_k + l_{ka}\tilde{\phi}_a)^2}{2t} - \sum_{k=V}^{E} \frac{(A_k + l_{ka}\tilde{\phi}_a - 2\pi m_k)^2}{2t}\right).
\end{aligned}
$$

Note that one cannot perform the induction step with the integration over $\tilde{\phi}_V$ as well. The reason for this is that for the induction step it is crucial that it does not make sense to define whether v_V is before or after any other vertex, since e_V does not belong to the maximal tree τ, in fact it does not even need to touch v_V. In particular, the integrand in (4.12) does not depend on $\tilde{\phi}_V$ at all! To see this, one only needs to shift all integrations $\tilde{\phi}_1, \ldots, \tilde{\phi}_{V-1}$ by $+\tilde{\phi}_V$. In all terms, the integration variables appear in the combination $\tilde{\phi}_a - \tilde{\phi}_b$ for any two different $a, b = 1, \ldots, V$. So either a and b are both not V, then nothing changes by this shift of integration, or one of a or b is equal to V. In this case the shift of the other one cancels the $\tilde{\phi}_V$, since both $\tilde{\phi}_a$ and $\tilde{\phi}_b$ appear with opposite sign. So, after this shift, $\tilde{\phi}_V$ occurs nowhere in the formula any more. Thus, we can perform the integration over $\tilde{\phi}_V$ trivially and obtain

$$
(4.9) \;=\; \sqrt{\frac{2\pi}{t}}^E \int_{\mathbb{R}^{V-1}} \frac{d\tilde{\phi}_1}{2\pi} \cdot \ldots \cdot \frac{d\tilde{\phi}_{V-1}}{2\pi} \sum_{m_V,\ldots,m_E\in\mathbb{Z}} \exp\left(-\sum_{k=1}^{E} \frac{(\tilde{A}_k + l_{ka}\tilde{\phi}_a)^2}{2t}\right)\Bigg|_{\tilde{\phi}_V=0} \qquad (4.13)
$$

where

$$
\tilde{A}_k \;:=\; \begin{cases} A_k & 1 \le k \le V-1 \\ A_k - 2\pi m_k & V \le k \le E \end{cases}. \qquad (4.14)
$$

To proceed, note that, since in every term in (4.13) the $\tilde{\phi}_a$ appear as pairs with opposite sign, the integrand is invariant under a simultaneous shift of all variables: $\tilde{\phi}_a \to \tilde{\phi}_a + c$. We use this fact to rewrite (4.13), by using the following technical Lemma

Lemma 4.3 *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a function with the symmetry*

$$
f(x_1 + c, \ldots, x_n + c) \;=\; f(x_1, \ldots, x_n) \qquad \textit{for all } c \in \mathbb{R}
$$

such that $x_1, \ldots, x_{n-1} \to f(x_1, \ldots, x_{n-1}, 0)$ is integrable. Then

$$
\int_{\mathbb{R}^{n-1}} d^{n-1}x\, f(x_1, \ldots, x_{n-1}, 0) \;=\; n \int_{\mathbb{R}^n} d^n x\, \delta(x_1 + \cdots + x_n)\, f(x_1, \ldots, x_n) \qquad (4.15)
$$

Proof: The proof is elementary. Write

$$
\begin{aligned}
&\int_{\mathbb{R}^{n-1}} dx_1, \dots dx_{n-1}\, f(x_1, \dots, x_{n-1}, 0) \\
&= \int_{\mathbb{R}^{n-1}} dx_1, \dots dx_{n-1}\, f\left(x_1 - \frac{\sum_{k=1}^{n-1} x_k}{n}, \dots, x_{n-1} - \frac{\sum_{k=1}^{n-1} x_k}{n}, -\frac{\sum_{k=1}^{n-1} x_k}{n}\right) \\
&= \int_{\mathbb{R}^{n}} dx_1, \dots dx_{n}\, f\left(x_1 - \frac{\sum_{k=1}^{n-1} x_k}{n}, \dots, x_{n-1} - \frac{\sum_{k=1}^{n-1} x_k}{n},\, x_n\right) \delta\left(x_n + \frac{\sum_{k=1}^{n-1} x_k}{n}\right)
\end{aligned}
$$

Now perform a coordinate transformation

$$
\begin{aligned}
\tilde{x}_k &:= x_k - \frac{\sum_{k=1}^{n-1} x_k}{n}, \qquad \text{for } k = 1, \dots, n-1 \\
\tilde{x}_n &:= x_n.
\end{aligned}
$$

We have

$$
\sum_{n=1}^{n-1} \tilde{x}_k = \frac{\sum_{k=1}^{n-1} x_k}{n}
$$

and get

$$
\begin{aligned}
&\int_{\mathbb{R}^{n-1}} dx_1, \dots dx_{n-1}\, f(x_1, \dots, x_{n-1}, 0) \\
&= \frac{1}{J} \int_{\mathbb{R}^{n}} d^n\tilde{x}\; f\left(\tilde{x}_1, \dots, \tilde{x}_{n-1},\, \tilde{x}_n\right)\; \delta(\tilde{x}_1 + \dots + \tilde{x}_n). \qquad (4.16)
\end{aligned}
$$

Here $J = \det\left(\partial\tilde{x}_k / \partial x_l\right)$ is the Jacobian matrix of the coordinate transform. It is given by

$$
J = \det\left[1 - \frac{1}{n}\begin{pmatrix} 1 1 \cdots 1 0 \\ 1 1 \cdots 1 0 \\ \vdots \vdots \ddots \vdots \vdots \\ 1 1 \cdots 1 0 \\ 0 0 \cdots 0 0 \end{pmatrix}\right],
$$

the determinant of which can easily computed to be $J = \frac{1}{n}$. Thus, with (4.16), the statement is proven.

We continue our analysis of the gauge-invariant overlap by using Lemma (4.3) to rewrite (4.13) to obtain

$$
(4.9) = V\sqrt{\frac{2\pi}{t}}^{E} \int_{\mathbb{R}^V} \frac{d\tilde{\phi}_1 \dots d\tilde{\phi}_V}{(2\pi)^{V-1}}\; \delta\left(\sum_{a=1}^{V} \tilde{\phi}_a\right) \sum_{m_V, \dots, m_E \in \mathbb{Z}} \exp\left(-\sum_{k=1}^{E} \frac{(\tilde{A}_k + l_{ka}\tilde{\phi}_a)^2}{2t}\right).
$$

Now we split the integrations over the $\tilde{\phi}_a$ from the $\tilde{A}_k$, in order to perform the integration. Because we are integrating over $\mathbb{R}^V$ and the integrand is holomorphic, we can now shift the $\tilde{\phi}_a$ also by complex amounts. This is necessary, since the $\tilde{A}_k$ are generically complex. A generic shift of the $\tilde{\phi}_a$ by complex numbers z_a looks like

$$\begin{aligned}
(4.9) &= V\sqrt{\frac{2\pi}{t}}^E \int_{\mathbb{R}^V} \frac{d\tilde{\phi}_1 \dots d\tilde{\phi}_V}{(2\pi)^{V-1}}\, \delta\left(\sum_{a=1}^V (\tilde{\phi}_a + z_a)\right) \\
&\quad \times \sum_{m_V,\dots,m_E \in \mathbb{Z}} \exp\left(-\sum_{k=1}^E \frac{(\tilde{A}_k + l_{ka}\tilde{\phi}_a + l_{ka}z_a)^2}{2t}\right) \\
&= V\sqrt{\frac{2\pi}{t}}^E \int_{\mathbb{R}^V} \frac{d\tilde{\phi}_1 \dots d\tilde{\phi}_V}{(2\pi)^{V-1}}\, \delta\left(\sum_{a=1}^V (\tilde{\phi}_a + z_a)\right) \\
&\quad \times \sum_{m_V,\dots,m_E \in \mathbb{Z}} \exp\left[-\sum_{k=1}^E \left(\frac{(l_{ka}\tilde{\phi}_a)^2}{2t} + \frac{l_{ka}\tilde{\phi}_a(l_{ka}z_a + \tilde{A}_k)}{t} + \frac{(l_{ka}z_a + \tilde{A}_k)^2}{2t}\right)\right] \\
&= V\sqrt{\frac{2\pi}{t}}^E \int_{\mathbb{R}^V} \frac{d\tilde{\phi}_1 \dots d\tilde{\phi}_V}{(2\pi)^{V-1}}\, \delta\left(u^T\tilde{\phi} + u^T z\right) \qquad (4.17) \\
&\quad \times \sum_{m_V,\dots,m_E \in \mathbb{Z}} \exp\left(-\frac{\tilde{\phi}^T l l^T \tilde{\phi}}{2t} - \frac{\tilde{\phi}^T l(l^T z + \tilde{A})}{t} - \frac{(l^T z + \tilde{A})^T (l^T z + \tilde{A})}{2t}\right).
\end{aligned}$$

In (4.17) we have expressed all variables in terms of vectors and matrices, since this will simplify the handling of the expressions a lot. The vectors u, $\tilde{\phi}$, z have length V, the vector $\tilde{A}$ has length E, and l is the $V \times E$ incidence matrix. The vector u is given by (4.5). The ^{T} means transpose.

The following Lemma will help us to simplify this formula.

Lemma 4.4 *Let l be the $V \times E$ incidence matrix of a connected graph γ with E edges and V vertices, and $u = (1\,1\,\cdots\,1)^T$ the vector of length V containing only ones. For any vector $\tilde{A} \in \mathbb{C}^E$ the set of equations*

$$\begin{aligned}
l(l^T z + \tilde{A}) &= 0 \\
u^T z &= 0
\end{aligned}$$

has exactly one solution in $\mathbb{C}^V$.

Proof: Rewrite the first of these equations as

$$l l^T z \;=\; -l\tilde{A}.$$

Because of (4.6), $-l\tilde{A}$ lies in the orthogonal complement of u: $-l\tilde{A} \in \{u\}^\perp$. Since the graph γ is connected, by Kirchhoff's theorem (4.1) the Kirchhoff-matrix ll^T is positive definite on $\{u\}^\perp$, hence invertible on this $V-1$-dimensional subspace of $\mathbb{C}^V$. Define the $V \times V$-matrix σ to be the inverse of ll^T on $\{u\}^\perp$, and zero on u:

$$\begin{aligned}
\sigma(ll^T)v \;=\; (ll^T)\sigma v = v \qquad &\text{for all } u^T v = 0 \\
\sigma u = 0.&
\end{aligned}$$

So, the set of solutions of $ll^T z = -l\tilde{A}$ is given by

$$z = -\sigma l\tilde{A} + \alpha u \qquad \alpha \in \mathbb{C}. \tag{4.18}$$

By the definition of σ, this means that

$$z = -\sigma l\tilde{A} \tag{4.19}$$

is the unique solution of both equations, which proves the Lemma.

Lemma 4.5 *With the conditions of Lemma 4.4, let z be the unique solution of $l(l^T z + \tilde{A}) = 0$ and $u^T z = 0$, i.e. $z = -\sigma l\tilde{A}$. Then*

$$-l^T\sigma l + 1_E = P_{\ker l}, \tag{4.20}$$

where 1_E is the $E \times E$ unit-matrix and $P_{\ker l}$ is the orthogonal projector onto the subspace $\ker l \subset \mathbb{C}^E$. In particular

$$l^T z + \tilde{A} = P_{\ker l}\tilde{A}. \tag{4.21}$$

Proof: Since

$$\ker l \oplus \text{img } l^T = 1_E, \tag{4.22}$$

The statement (4.20) can be rephrased as follows:

$$l^T\sigma l = P_{\text{img } l^T}, \tag{4.23}$$

which is the projector onto the image of l^T. Let $v \in \text{img } l^T$, so $v = l^T w$ for some $w \in \mathbb{C}^V$. Even more, since $l^T u = 0$, we even can choose w to be orthogonal to u: $w \in \{u\}^\perp$. Then

$$l^T\sigma l\, v = l^T\sigma(ll^T)w = l^T w = v,$$

since by definition σ is the inverse of ll^T on $\{u\}^\perp$.

Let, on the other hand, $v \in \{\text{img } l^T\}^\perp = \ker l$. Then

$$l^T\sigma l\, v = 0$$

trivially. Thus, $l^T\sigma l$ leaves vectors in img l^T invariant and annihilates vectors from the orthogonal complement of img l^T. Hence $l^T\sigma l = P_{\text{img } l^T}$, from which it follows that

$$-l^T\sigma l + 1_E = P_{\ker l}.$$

This was the first claim, the second one

$$l^T z + \tilde{A} = P_{\ker l}\tilde{A}.$$

follows immediately.

The Lemmas 4.4 and 4.5 enable us to rewrite (4.17) as

$$(4.9) = V\sqrt{\frac{2\pi}{t}}^{E} \int_{\mathbb{R}^V} \frac{d\tilde{\phi}_1 \dots d\tilde{\phi}_V}{(2\pi)^{V-1}}\, \delta\left(u^T\tilde{\phi}\right) \times \sum_{m_V,\dots,m_E\in\mathbb{Z}} \exp\left(-\frac{\tilde{\phi}^T l l^T \tilde{\phi}}{2t} - \frac{\tilde{A}^T P_{\ker l}\tilde{A}}{2t}\right). \tag{4.24}$$

We can now finally evaluate the gauge integrals in (4.24) with the help of Kirchhoff's theorem. Since the delta-function in the integrand of (4.24) assures that we only integrate over the orthogonal complement of u, instead of $\mathbb{R}^V$, and Kirchhoff's theorem 4.1 assures that the Kirchhoff-matrix ll^T is positive definite there, we can immediately evaluate the integral:

$$\int_{\mathbb{R}^V} \frac{d\tilde{\phi}_1 \cdots d\tilde{\phi}_V}{(2\pi)^{V-1}}\, \delta\left(u^T\tilde{\phi}\right)\, \exp\left(-\frac{\tilde{\phi}^T l l^T \tilde{\phi}}{2t}\right) = \sqrt{\frac{t}{2\pi}}^{V-1} \frac{1}{\sqrt{\prod_{a=2}^V \mu_a}} = \frac{1}{\sqrt{G\,V}}\sqrt{\frac{t}{2\pi}}^{V-1} \tag{4.25}$$

where $\mu_2, \dots, \mu_V$ are the nonzero eigenvalues of the Kirchhoff-matrix, and G is the number of different possible maximal trees in the graph γ. With this, the gauge-invariant coherent state can be written as

$$(4.9) \;=\; \sqrt{\frac{V}{G}}\sqrt{\frac{2\pi}{t}}^{E-V+1} \sum_{m_V,\dots,m_E\in\mathbb{Z}} \exp\left(-\frac{(A-2\pi m)^T P_{\ker l}(A-2\pi m)}{2t}\right)$$

where $A = z - \phi$ is the vector containing $A_k = z_k - \phi_k$ in its k-th component, and m being the vector containing 0 in the first $V-1$ components and $m_V, \dots, m_E$ in the last $E-V+1$ components.

As already stated, the kernel of l is $E-V+1$-dimensional. Let $l_1, \dots, l_{E-V+1}$ be an orthonormal basis of $\ker l \subset \mathbb{R}^E$. Define

$$z_\nu^{gi} \;:=\; l_\nu^T z, \qquad \phi_\nu^{gi} \;:=\; l_\nu^T \phi, \qquad m_\nu^{gi} \;:=\; l_\nu^T m. \tag{4.26}$$

With this and $P_{\ker l} = \sum_{\nu=1}^{E-V+1} l_\nu l_\nu^T$, we get our final formula

$$\Psi_{[z]}^t(\phi) = \sqrt{\frac{V}{G}}\sqrt{\frac{2\pi}{t}}^{E-V+1} \sum_{m_V,\dots,m_E\in\mathbb{Z}} \exp\left(-\sum_{\nu=1}^{E-V+1} \frac{(z_\nu^{gi} - \phi_\nu^{gi} - 2\pi m_\nu^{gi})^2}{2t}\right). \tag{4.27}$$

The gauge-invariant coherent state only depends on the z_ν^{gi} and ϕ_ν^{gi}, which are gauge-invariant combinations of the z_k and ϕ_k. That the linear combinations (4.26) are gauge-invariant, is clear from the construction, but one can immediately see this from the following: Perform a gauge-transformation, which shifts the ϕ_k by $l_{ka}\tilde{\phi}_a$. So, in matrices, one has $\phi \to \phi + l^T\tilde{\phi}$. Thus,

$$\phi_\nu^{gi} \;=\; l_\nu^T\phi \;\longrightarrow\; l_\nu^T(\phi + l^T\tilde{\phi}) \;=\; l_\nu^T\phi \;+\; l_\nu^T l^T\tilde{\phi} \;=\; l_\nu^T\phi \;=\; \phi_\nu^{gi},$$

where $l_\nu \in \ker l$ has been used, from which it follows that $l l_\nu = 0$, so $l_\nu^T l^T = 0$. Thus, the linear combinations of ϕ in ϕ_ν^{gi} are all gauge-invariant. The same holds true, of course, for the z_ν^{gi} and m_ν^{gi}. So, the coherent states depend only on gauge-invariant combinations of ϕ, which was clear from the beginning, but can now be seen explicitly. Note that the basis $\{l_\nu\}_{\nu=1}^{N-V+1}$ is, of course, not unique, but can be replaced by any other basis $l'_\nu = R_{\nu\mu} l_\mu$ with $R \in O(E-V+1)$.

Compare the formula for the gauge-invariant coherent state (4.27) with the formula for the gauge-variant coherent states on E edges (4.8). Up to a factor of $(V/G)^{1/2}$, the similarity is striking. One could be led to the conclusion that gauge-invariant coherent states are nothing but gauge-variant coherent states, only depending on gauge-invariant quantities. The fact that the gauge-invariant configuration space is diffeomorphic to $U(1)^{E-V+1}$, supports this guess.

However, this is not true. The reason is that the summation variables $m_V, \ldots, m_E$ are placed in wrong linear combinations in the formula. In particular, a gauge-invariant state is *not*

$$\Psi_{[z]}^t(\phi) \neq \sqrt{\frac{V}{G}} \sqrt{\frac{2\pi}{t}}^{E-V+1} \sum_{m_1^{gi},\ldots,m_{E-V+1}^{gi} \in \mathbb{Z}} \exp\left(- \sum_{\nu=1}^{E-V+1} \frac{(z_\nu^{gi} - \phi_\nu^{gi} - 2\pi m_\nu^{gi})^2}{2t} \right)$$

$$= \sqrt{\frac{V}{G}} \psi_{z^{gi}}^t(\phi^{gi}). \tag{4.28}$$

Of course, from the form (4.27) one cannot deduce a priori that the m_ν^{gi} could not, probably, be reordered in a way, maybe by an intelligent choice of l_ν and/or suitable shifting of summations, such that a form like (4.28), possibly with different t for different variables, could be possible. But already at simple examples like the 3-bridge graph show that this cannot be done. It could be, if one is lucky (in particular, on the 2-bridge graph), but generically a gauge-invariant coherent state is no complexifier coherent state depending on gauge-invariant variables.

4.5 Peakedness of gauge-invariant coherent states

In this chapter, we will shortly investigate the peakedness properties of the gauge-invariant coherent states. In particular, we will show that they are peaked on gauge-invariant quantities. Let γ be a graph with E edges. Then, a complexifier coherent state is then labeled by E complex numbers $z_1, \ldots, z_E$ and a semiclassicality parameter $t > 0$. Such a state is given by

$$\psi_z^t(\phi) \;=\; \sqrt{\frac{2\pi}{t}}^E \sum_{m_1,\ldots,m_E \in \mathbb{Z}} \exp\left(- \sum_{k=1}^{E} \frac{(z_k - \phi_k - 2\pi m_k)^2}{2t} \right). \tag{4.29}$$

The corresponding gauge-invariant coherent states, obtained by applying the projector onto the gauge-invariant sub-Hilbert-space, has, in the last section, been shown to be

$$\Psi_{[z]}^t(\phi) \;=\; \sqrt{\frac{V}{G}} \sqrt{\frac{2\pi}{t}}^{E-V+1} \sum_{m_V,\ldots,m_E \in \mathbb{Z}} \exp\left(- \sum_{\nu=1}^{E-V+1} \frac{(z_\nu^{gi} - \phi_\nu^{gi} - 2\pi m_\nu^{gi})^2}{2t} \right).$$

Here G is the number of different possible maximal trees is the graph γ and $\phi_\nu^{gi} = l_\nu^T \phi$, where $l_1, \ldots, l_{E-V+1}$ is an orthonormal base for the kernel $\ker l \subset \mathbb{R}^E$ of the incidence matrix l of γ. Also, $z_\nu^{gi} = l_\nu^T z$ and $m_\nu^{gi} = l_\nu^T m$, where m is the vector containing zeros in the first $V-1$ entries, and m_V to m_E in the last $E-V+1$ entries.

The inner product between two gauge-invariant coherent states $\Psi_{[w]}^t$ and $\Psi_{[z]}^t$ is, as one can easily calculate, given by

$$\left\langle \Psi_{[w]}^t \middle| \Psi_{[z]}^t \right\rangle \;=\; \sqrt{\frac{V}{G}}\sqrt{\frac{\pi}{t}}^{E-V+1} \sum_{m_V,\ldots,m_E\in\mathbb{Z}} \exp\left(-\sum_{\nu=1}^{E-V+1} \frac{(\bar{w}_\nu^{gi} - z_\nu^{gi} - 2\pi m_\nu^{gi})^2}{t}\right). \tag{4.30}$$

With $z_k = \phi_k - ip_k$, i.e. by splitting the phase-space points into configuration- and momentum variables, one immediately gets a formula for the norm of a gauge-invariant coherent state:

$$\left\|\Psi_{[z]}^t\right\|^2 \;=\; \sqrt{\frac{V}{G}}\sqrt{\frac{\pi}{t}}^{E-V+1} \sum_{m_V,\ldots,m_E\in\mathbb{Z}} \exp\left(4\sum_{\nu=1}^{E-V+1} \frac{(p_\nu^{gi} - \pi i m_\nu^{gi})^2}{t}\right). \tag{4.31}$$

Note that there is, apart from $m = 0$, no combination of $m_V, \ldots, m_E$ such that the corresponding $m_\nu^{gi} = 0$ for all $\nu = 1, \ldots, E-V+1$. If there is one such combination, there are infinitely many of these combinations, hence infinitely many equally large terms. So, if there were, then the sum in (4.30) would not exist at all. But we know that the sum in (4.30) is absolutely convergent, so there is no such combination.

What we just said is equivalent to saying that

$$P_{\ker l}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_V \\ \vdots \\ m_E \end{pmatrix} \;\neq\; 0 \qquad \text{for all } m_V, \ldots m_E \in \mathbb{Z},$$

which is, of course, due to the fact that the last $E-V+1$ components correspond, by construction, to the gauge-invariant directions on $U(1)^E$. In the limit of $t \to 0$, the norm of a gauge-invariant coherent state (4.31) can be written as

$$\begin{aligned}\left\|\Psi_{[z]}^t\right\|^2 &\leq \sqrt{\frac{V}{G}}\sqrt{\frac{\pi}{t}}^{E-V+1} \sum_{m_V,\ldots,m_E\in\mathbb{Z}} \exp\left(4\sum_{\nu=1}^{E-V+1} \frac{(p_\nu^{gi})^2 - \pi^2 (m_\nu^{gi})^2}{t}\right) \\ &= \sqrt{\frac{V}{G}}\sqrt{\frac{\pi}{t}}^{E-V+1} \exp\left(4\sum_{\nu=1}^{E-V+1}\frac{(p_\nu^{gi})^2}{t}\right) \sum_{m_V,\ldots,m_E\in\mathbb{Z}} \exp\left(-4\pi^2 \sum_{\nu=1}^{E-V+1} \frac{m^T P_{\ker l} m}{t}\right)\end{aligned} \tag{4.32}$$

Define

$$K \;:=\; \min_{\|m\|=1} \|P_{\text{ker}} m\| \;>\; 0.$$

With this, $m^T P_{\text{ker}} m \;\geq\; K^2 \|m\|^2$, so we get

$$\begin{aligned}
\sum_{m_V,\ldots,m_E \in \mathbb{Z}} \exp\left(-4\pi^2 \sum_{\nu=1}^{E-V+1} \frac{m^T P_{\text{ker}l} m}{t}\right) &\leq \sum_{m_V,\ldots,m_E \in \mathbb{Z}} \exp\left(-4\pi^2 K^2 \frac{\|m\|^2}{t}\right) \\
&= \left[\sum_{n\in\mathbb{Z}} \exp\left(\frac{-4\pi^2 K^2}{t} n^2\right)\right]^{E-V+1} \qquad (4.33) \\
&= 1 \,+\, O(t^\infty).
\end{aligned}$$

Thus, we can write

$$\left\|\Psi^t_{[z]}\right\|^2 \;=\; \sqrt{\frac{V}{G}}\sqrt{\frac{\pi}{t}}^{E-V+1} \sum_{m_V,\ldots,m_E \in \mathbb{Z}} \exp\left(4 \sum_{\nu=1}^{E-V+1} \frac{(p^{gi}_\nu)^2}{t}\right)(1 + O(t^\infty)).(4.34)$$

For the inner product between complexifier coherent states, one has

$$\left\langle \psi^t_w \middle| \psi^t_z \right\rangle \;=\; \left\langle \psi^t_0 \middle| \psi^t_{z-\bar{w}} \right\rangle, \qquad (4.35)$$

as can be readily deduced from the explicit formula of the inner product between two complexifier coherent states. This is also true for the gauge-invariant coherent states, which have

$$\left\langle \Psi^t_{[w]} \middle| \Psi^t_{[z]} \right\rangle \;=\; \left\langle \Psi^t_{[0]} \middle| \Psi^t_{[z-\bar{w}]} \right\rangle. \qquad (4.36)$$

This can either be deduced by applying the gauge-projector onto (4.35), or directly from formula (4.30).

So, in order to show that the overlap of two gauge-invariant coherent states, labeled by $[z]$ and $[w]$, is peaked at $[z] = [w]$, one only has to show that the overlap between a state labeled by $[z]$ and $\Psi^t_{[0]}$ is peaked at $[z] = [0]$. With (4.34) and $z = \phi - ip$, we get

$$\begin{aligned}
\frac{\left\langle \Psi^t_{[0]} \middle| \Psi^t_{[z]} \right\rangle}{\left\|\Psi^t_{[0]}\right\| \; \left\|\Psi^t_{[z]}\right\|} &= \sum_{m_V,\ldots,m_E \in \mathbb{Z}} \exp\left(-\sum_{\nu=1}^{E-V+1} \frac{(\phi^{gi}_\nu - ip^{gi}\nu + 2\pi m^{gi}_\nu)^2}{t} - \sum_{\nu=1}^{E-V+1} \frac{2(p^{gi}_\nu)^2}{t}\right) \\
&\quad \times (1 + O(t^\infty)) \\
&= \sum_{m_V,\ldots,m_E \in \mathbb{Z}} \exp\left(-\sum_{\nu=1}^{E-V+1} \frac{(\phi^{gi}_\nu - 2\pi m^{gi}_\nu)^2 + (p^{gi}_\nu)^2}{t} + 2i \frac{p^{gi}_\nu (\phi^{gi}_\nu - 2\pi m^{gi}_\nu)}{t}\right) \\
&\quad \times (1 + O(t^\infty)).
\end{aligned}$$

If the ϕ_ν^{gi} are close to zero, then the term with all $m_\nu^{gi} = 0$, which corresponds to all $m_k = 0$, is significantly larger than the other terms. So this can, with similar arguments as in (4.33), be further simplified into

$$\frac{\left\langle \Psi^t_{[0]} \middle| \Psi^t_{[z]} \right\rangle}{\left\| \Psi^t_{[0]} \right\| \, \left\| \Psi^t_{[z]} \right\|} = \exp\left(- \sum_{\nu=1}^{E-V+1} \frac{(\phi_\nu^{gi})^2 + (p_\nu^{gi})^2}{t} + 2i \frac{p_\nu^{gi} \phi_\nu^{gi}}{t} \right) (1 + O(t^\infty)) \tag{4.37}$$

This approaches 1 if the gauge-invariant quantities ϕ^{gi} and p^{gi} are close to zero, but as soon as the gauge-invariant quantities are away from zero, the expression becomes tiny, due to the tiny t. It follows that the overlap is peaked at gauge-invariant quantities.

5 Summary and conclusion

This is the first of two articles concerning the gauge-invariant coherent states for Loop Quantum Gravity. In this one, we have replaced the gauge-group $G = SU(2)$ of LQG by the much simpler $G = U(1)$, the case $G = U(1)^3$, which is also of interest for LQG, follows immediately. We have investigated the gauge-invariant coherent states, in particular we have computed their explicit form and their overlap. The results found are very encouraging: While the complexifier coherent states are peaked on points in the kinematical phase space, which contains gauge information, the gauge-invariant coherent states, which are labeled by gauge-equivalence classes, are also sharply peaked on these. In particular, the overlap between two gauge-invariant coherent states labeled with different gauge orbits tends to zero exponentially fast as the semiclassicality parameter t tends to zero. Even more, it could be shown that the overlap is actually a Gaussian in the gauge-invariant variables.

This shows the good semiclassical properties of these states: As t tends to zero, different states become approximately orthogonal very quickly, suppressing the quantum fluctuations between them. Also, the expectation values of operators corresponding to gauge-invariant kinematical observables (such as volume or area) are approximated well, which immediately follows from the corresponding properties of the gauge-variant CCS states.

This shows that the gauge-invariant coherent states are in fact useful for the semiclassical analysis of the gauge-invariant sector of LQG, and is the first step on the road to *physical* coherent states.

Apart from the nice semiclassical properties, the computation has revealed an explicit connection between the gauge-invariant sector and the graph topology. In particular, the formula for the gauge-invariant coherent states on a graph γ contains the incidence matrix l of γ. In contrast, the CCS are simply a product of states on each edge of the graph, hence have no notion of which edges are connected to each other and which are not, while the gauge-invariant coherent states explicitly contain information about the graph topology. This is simply due to the fact that the set of gauge-invariant degrees of freedom depend on the graph topology and can be computed by graph-theoretic methods.

While the results for $G = U(1)$ are quite encouraging, the case of ultimate interest for LQG is $G = SU(2)$, which is much more complicated. We will address this topic in the following article, which will deal with this issue and try to establish as much results as possible from $U(1)$, where the problem could be solved completely and analytically, also for $SU(2)$.

Acknowledgements

BB would like to thank Hendryk Pfeiffer for the discussions about gauge-invariant functions and cohomology. Research at the Perimeter Institute for Theoretical Physics is supported by the Government of Canada through NSERC and by the Province of Ontario.

A Cohomology with values in abelian groups

In the following we will write down the definition for singular cohomology with values in an abelian group. This will allow for a compact notation of the gauge-invariant Hilbert space. In particular, we will characterize the cohomology spaces in question to arrive at a better understanding what to expect, when computing the gauge-invariant coherent states on graphs and their overlaps. Note that we will employ, for brevity, the notation

$$A^B \;:=\; \{f : B \to A \text{ any map}\} \tag{A.1}$$

for the set of maps from any set B to any set A.

Consider a CW complex K, i.e. a topological space that is successively built up of n-cells (n-dimensional closed balls), such that the intersection of two cells is a collection of lower-dimensional sub-cells, and around each point there is a neigbourhood that contains finitely many cells. In particular, any graph in Σ is a CW complex of dimension 1, i.e. consisting only of 1-cells (the edges), that intersect at the 0-cells (the vertices).

Let K^n be the set of all n-cells in the CW complex K. Let G be an abelian group, then define $C^n(K,G)$ to be the set G^{K^n}, i.e. the set of all maps from K^n to G. Then $C^n(K,G)$ is obviously an abelian group, simply by defining the group multiplication pointwise. This group is obviously homomorphic to $G^{|K^n|}$.

We then define a chain by

$$\{1\} \;\xrightarrow{\delta}\; C^0(K,G) \;\xrightarrow{\delta}\; C^1(K,G) \;\xrightarrow{\delta}\; C^2(K,G) \;\xrightarrow{\delta}\; \ldots \tag{A.2}$$

where $\delta : C^n(K,G) \to C^{n+1}(K,G)$ is defined by the following rule: Let $f : K^n \to G$ be an element of $C^n(K,G)$. Then, for an $n+1$-cell c we define

$$\delta f(c) \;:=\; f(v_1)^{\sigma_1} \cdot \ldots \cdot f(v_{|K^n|})^{\sigma_{|K^n|}}, \tag{A.3}$$

where $v_1, \ldots v_{|K^n|}$ are all n-cells and the factors σ_k are defined to be $+1$ if v_k is part of the boundary of c and the orientation of v_k is the same as the induced one from

c, -1 if v_k is in the boundary of c but the induced orientation from c and the given one on v_k differ by a sign, 0 if v_k is not part of the boundary of c.

Note that δ is a group homomorphism, which follows from the abeliness of G. Hence, for each n, both sets $\ker\ \delta : C^n(K,G) \to C^{n+1}(K,G)$ and $\operatorname{img}\delta : C^{n-1}(K,G) \to C^n(K,G)$ are subgroups of $C^n(K,G)$, where the kernel of a group homomorphism is defined to be the set of all elements being mapped to the unit element.

One can explicitly check that with this definition, that the map

$$\delta\delta : C^n(K,G) \longrightarrow C^{n+2}(K,G)$$

maps every $C^n(K,G)$ to the unit element in $C^{n+2}(K,G)$. It follows that even $\operatorname{img}\delta : C^{n-1}(K,G) \to C^n(K,G)$ is a subgroup of $\ker\ \delta : C^n(K,G) \to C^{n+1}(K,G)$. Thus, one can define the quotients

$$H^n(K,G) \ := \ \frac{\ker\ \delta : C^n(K,G) \to C^{n+1}(K,G)}{\operatorname{img}\delta : C^{n-1}(K,G) \to C^n(K,G)},$$

which is called the *n-th cohomology group of K with values in G*. As the name suggests, this is of course also an abelian group.

The definition above is fairly general, but we will now see what it means for the specific case of the CW complex being an oriented graph γ (with the orientations of the vertices all being set to the number $+1$). We keep the abelian group G arbitrary for the moment, having in mind the application to $G = U(1)$ or $G = U(1)^3$ lateron.

Let us consider a graph γ, consisting of a set of edges $E(\gamma)$ and vertices $V(\gamma)$. The chain in (A.2) is then simply

$$\{1\} \stackrel{\delta}{\longmapsto} G^{V(\gamma)} \stackrel{\delta}{\longmapsto} G^{E(\gamma)} \stackrel{\delta}{\longmapsto}; \{1\}.$$

The only nontrivial map is $\delta : G^{V(\gamma)} \to G^{E(\gamma)}$. For every edge $e \in E(\gamma)$, $b(e)$ and $f(e)$ are called the beginning- and endpoint of the edge, and are by construction both elements of $V(\gamma)$. So let $k : V(\gamma) \to G$ be an element of $G^{V(\gamma)}$. Then the definition of δ given above implies

$$(\delta k)_e \ = \ k_{b(e)}\, k_{f(e)}^{-1}, \tag{A.4}$$

so δk is a map from $E(\gamma)$ to G, that is an element of $G^{E(\gamma)}$. The only nontrivial cohomology groups we can form are then

$$H^0(\gamma,G) = \frac{\ker\ \delta : G^{V(\gamma)} \to G^{E(\gamma)}}{\operatorname{img}\delta : \{1\} \to G^{V(\gamma)}} \ = \ \ker\ \delta : G^{V(\gamma)} \to G^{E(\gamma)}, \tag{A.5}$$

$$H^1(\gamma,G) = \frac{\ker\ \delta : G^{E(\gamma)} \to \{1\}}{\operatorname{img}\delta : G^{V(\gamma)} \to G^{E(\gamma)}} \ = \ \frac{G^{E(\gamma)}}{\operatorname{img}\delta : G^{V(\gamma)} \to G^{E(\gamma)}}. \tag{A.6}$$

These two groups have nice interpretations in terms of the graph topology, which are stated by the following lemma:

Lemma A.1 *Let γ be a graph (connected, oriented, finitely many edges, embedded in a 3-manifold Σ). Then, for any abelian group G, we have*

$$H^0(\gamma, G) \simeq G, \tag{A.7}$$
$$H^1(\gamma, G) \simeq Hom\ (\pi_1(\gamma), G), \tag{A.8}$$

Loosely speaking, $H^0(\gamma, G)$ counts the connected parts of γ, and $H^1(\gamma, G)$ counts the numbers of "holes" in γ.

Proof: The proof is quite standard, but we will still repeat it here.

By (A.4) and (A.5), we see that $H^0(\gamma, G)$ consists of all maps k from $V(\gamma)$ to G, such that, for every edge e, $k_{b(e)}\, k_{f(e)}^{-1} = 1$. Since the graph is connected, this is equivalent to saying that the map k assigns to each vertex $v \in V(\gamma)$ the same element in G:

$$k_v \;=\; h \quad \text{for some } h \in G \text{ and all } v \in V(\gamma).$$

The group of all such maps is then clearly equivalent to the group G itself, since the graph γ is connected. So we have

$$H^0(\gamma, G) \;\simeq\; G.$$

To show the second part of (A.7), consider a maximal tree τ in the graph γ. A maximal tree is a subgraph such that each vertex of γ is also contained in τ (i.e. $V(\gamma) = V(\tau)$), and the graph τ contains no closed loops. Call all edges in γ that are not in τ leaves. Maximal trees exist for all graphs, although they are far from unique. The number of leaves in a graph, though, is independent from the choice of τ.

To compute $H^1(\gamma, G)$, we have to compute the orbits of the subgroup $\delta(G^{V(\gamma)}) \subset G^{E(\gamma)}$. We do this by showing that, to each $h \in G^{E(\gamma)}$, we can apply an element of $\delta(G^{V(\gamma)}$, such that the result is an element $\tilde{h} \in G^{E(\gamma)}$ such that $\tilde{h}_e = 1$ for all $e \in E(\tau)$. In short, we show that one can gauge fix the group elements on the edges belonging to the tree τ to 1. The remaining distribution of elements $\tilde{h}_e$ for leaves e is unique, due to the fact that the group G is abelian.

Consider an element h of $G^{E(\gamma)} = C^1(\gamma, G)$, i.e. a distribution $(h_{e_1}, \ldots, h_{e_E})$ of elements in G among the edges in $E(\gamma)$. Construct an element $k \in V(\gamma)$ by the following method: Choose a vertex v in $V(\gamma)$. For each other vertex $w \in V(\gamma)$, there is a unique path from w to v along edges in τ, since τ contains no loops. So, in order to get from w to v, one has to go, say, along edges $e_1, \ldots e_n$, either parallel or antiparallel to the orientation of the e_i. Define the k_w to be the product

$$k_w \;=\; h_{e_1}^{\pm 1} h_{e_2}^{\pm 1} \cdots h_{e_n}^{\pm 1}, \tag{A.9}$$

where the element h_{e_i} is contained in the product, if the path from w to v is parallel to the orientation of e_i. If the path is antiparallel, then take $h_{e_i}^{-1}$ instead.

Thus, an element $k \in G^{V(\gamma)}$ is defined. Now consider the product $\tilde{h} := \delta k \cdot h$. It is quite easy to see that the element $\tilde{h}$ assigns $1 \in G$ to each $e \in E(\tau)$: consider an $e \in E(\tau)$. The path from $f(e)$ to v passes through $b(e)$, or the other way round. Assume the first to be the case, the other case works analogously. We have then

$$k_{f(e)} \;=\; h_e\, k_{b(e)},$$

since the path from $f(e)$ goes against the orientation of e to $b(e)$, and then is identical to the way from $b(e)$ to v, since τ contains no loops. So

$$\tilde{h}_e \;=\; k_{b(e)}\, h_e\, k_{f(e)}^{-1} \;=\; 1.$$

Thus, we have shown, the orbit of each element $h \in G^{E(\gamma)}$ under the action of the subgroup $\delta(G^{V(\gamma)})$ contains an element $\tilde{h}$ such that only the elements assigned to the leaves in γ are potentially different from $1 \in G$. One can also see that the only element in $\delta(G^{V(\gamma)})$ that leaves the distribution of 1 along the edges of $E(\tau)$ unchanged, is an element $k \in \ker\delta$, i.e. such that $k_v = h$ for some $h \in G$ and all $v \in V(\gamma)$. The multiplication with δk leaves $G^{E(\gamma)}$ invariant, since G is abelian. We thus see that the element of $\tilde{h}$ is unique for each $h \in E(\gamma)$, hence does not depend on the choice of the vertex v. This shows that each orbit in $G^{E(\gamma)}$ under the action of $\delta(G^{V(\gamma)})$ determines uniquely a distribution of group elements in G among the leaves of γ.

Since τ contains no loops, it is contractible. Consider the flower graph $\tilde{\gamma}$ that one obtains by contracting the tree τ to a point. This graph contains just one vertex V and a number of edges, all starting and ending at V, corresponding to the number of leaves of γ. Note that $H^1(\tilde{\gamma}, G) = E(\tilde{\gamma})^G$. From this and our considerations above it follows that there is a natural group isomorphism between $H^1(\gamma, G) \simeq H^1(\tilde{\gamma}, G)$. It is clear that the first fundamental group $\pi_1(\tilde{\gamma})$ is freely generated by the elements of $E(\tilde{\gamma})$. In particular, $H^1(\tilde{\gamma}, G) \simeq \mathrm{Hom}(\pi_1(\tilde{\gamma}), G)$.

Since τ contains no loops, the tree is contractible, hence $\tilde{\gamma}$ is a retraction of γ. In particular, both graphs are homotopy equivalent. Since the first fundamental group is a homotopy invariant, we conclude

$$H^1(\gamma, G) \;\simeq\; \mathrm{Hom}(\pi_1(\gamma), G).$$

In particular, $H^1(\gamma, G) \simeq G^L$, where L is the number of leaves in γ (which is independent of the choice of the maximal tree τ).

A.1 Gauge-invariant functions

The notion of gauge-invariant cylindrical functions fits nicely into the framework of cohomology. Remember that a gauge-variant function on a graph γ is determined via (2.2) by a function of a number of copies of the gauge group G:

$$\tilde{f}: \qquad \underbrace{G \times \cdots \times G}_{\text{one for each edge in } E(\gamma)} \qquad \to \mathbb{C}$$

that is square-integrable with respect to the product Haar-measure $d\mu_H^{\otimes|E(\gamma)|}$. These function constitute the Hilbert space $\mathcal{H}_\gamma$, and with the notions of the previous sections, we identify this space to be

$$\mathcal{H}_\gamma \;\simeq\; L^2\big(C^1(\gamma, G),\; d\mu_H^{\otimes|E(\gamma)|}\big). \tag{A.10}$$

The gauge transformed $\tilde{f}$ is determined by letting the gauge group G act on every vertex $v \in V(\gamma)$ via (2.7):

$$\alpha_{k_{v_1},\ldots,k_{v_V}} \tilde{f}\,\big(h_{e_1}, \ldots, h_{e_E}\big) \;=\; \tilde{f}\big(k_{b(e_1)}^{-1} h_{e_1} k_{f(e_1)},\; \ldots\;, k_{b(e_E)}^{-1} h_{e_E} k_{f(e_E)}\big),$$

where $b(e)$ and $f(e)$ are the vertices sitting at the beginning and the end of the edge e respectively.

Not only do we recognize the gauge transformation group as the space $G^{V(\gamma)} = C^0(\gamma, G)$ from the previous section, one can see readily the connection between the gauge transformation α and the coboundary operator δ:

$$(\alpha_{g_1,\ldots,g_V}\tilde{f})(h_1,\ldots,h_E) \;=\; \tilde{f}\big(\delta(g_1,\ldots,g_V)\cdot(h_1,\ldots,h_E)\big),$$

where $\cdot$ means group multiplication in $C^1(\gamma, G) = G^{E(\gamma)}$. So, the gauge-invariant functions on the graph γ are just the functions on the group $G^{E(\gamma)}$ that are invariant under the action of $\delta(G^{V(\gamma)})$. We conclude that the gauge-invariant functions coincide with the functions on the first cohomology class

$$\mathcal{PH}_\gamma \;\simeq\; L^2\big(H^1(\gamma, G),\; d\mu\big), \tag{A.11}$$

where the measure $d\mu$ is the quotient measure of $d\mu_H^{\otimes|E(\gamma)|}$ under the action of the gauge transformation group $G^{V(\gamma)}$, which, since $H^1(\gamma, G)$ is a group for abelian G, can be identified with the normalized Haar measure on $H^1(\gamma, G)$.

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